Complex Transforms

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COMPLEX TRANSFORMS

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ABSTRACT

In this research paper**,** motivated by the concept of complex hypercube, a novel class of Complex Hadamard Matrices are proposed. Based on such class of matrices, a novel transform, called Complex Hadamard Transform is discussed. In the same spirit of this transform, other complex transforms such as Complex Haar transform are proposed. It is expected that these novel complex transforms will find many applications. Also, the associated complex valued orthogonal functions are of theoretical interest.

1. **Introduction**:

Homosapien civilizations, across the planet innovated positional number systems to conduct day-to-day life. Napier discovered the concept of logarithm that enabled him to convert the operation of multiplication of large numbers to additions of small numbers in some sense. This discovery led to the concept of "transform" in primitive form. Discoveries in algebra led to the concept of a function, more particularly periodic function. Fourier, in his studies on heat conduction problems proposed the representation of a periodic function in terms of trigonometric functions. The coefficients of so called Fourier series were computed explicitly. To extend the idea to aperiodic functions, Fourier devised the so called "Fourier transform" of real valued signals as well as complex valued signals. The concept was generalized by Laplace to more general class of functions using the so called Laplace transform. In the case of design and analysis of linear time invariant systems, Fourier / Laplace transform enabled deriving very elegant and powerful results. Researchers naturally questioned the representability of signals that are defined on a finite support in terms of other interesting finite collection of signals. In that effort, various families of orthonormal basis functions were discovered. It has been shown that functions / signals in certain function spaces (i.e. $L^p - spaces$) can be expressed in terms of orthonormal basis functions.

With origins in solution of linear system of equations, linear algebra was established as an important subject with applications in science, engineering and other fields of human endeavor. The concepts such as vector space, linear independence, dimension, basis were provided sound logical / mathematical footing. They led to the concept of orthonormal basis vectors. With the important result that any vector can be expressed uniquely using a set of orthonormal basis vectors, researchers focused their efforts on finding various such basis vectors. Sylvester and Hadamard independently discovered matrices whose rows / columns form an orthonormal basis. Such basis vectors also lead to orthonormal basis functions on a finite support.

This research paper is organized as follows. In Section 2, real Hadamard matrix and real Hadamard transform are reviewed. In Section 3, using complex hypercube, complex Hadamard matrix is defined and the associated complex Hadamard transform is defined. In Section 4, other complex transforms are briefly defined. The research paper concludes in Section 5.

2. Real Hadamard Matrices : Real Hadamard Transform :

Linear algebra as a field of human endeavour found many applications. Mathematicians conceived of finite dimensional linear operators, i.e. matrices with special structure such as Toeplitz, Hankel, Hilbert , Vandermonde matrices. In fact L-matrices were proposed in [7]. In a well defined sense, such matrices naturally arise in many applications in science, engineering and other areas. Sylvester as well as Hadamard independently conceived of one such matrix with special structure, now called the Hadamard matrix. Each of the elements of the Hadamard matrix belong to the set { +1, -1 }. Thus, the rows / columns of the Hadamard matrix (square matrix) are the corners of the symmetric, unit hypercube i.e. Mathematically, this set is specified precisely as follows:

$$S = \{ \overline{X} = (x_i, x_2, x_3, ..., x_N) : x_i = \pm 1 \, for 1 \leq i \leq N \} .$$

Formally, we have the following definition [4] :

Definition: A Hadamard matrix of order 'm', denoted by H_m , is an m x m matrix of +1's and -1's such that

$$H_m \, H_m^T = m \; I_m$$

where I_m is the m x m identity matrix. This definition is equivalent to saying that any two rows of H_m are orthogonal.

In view of the above definition and Lemma 1, we have the following interesting result

Lemma 1 : Hadamard matrices of odd order donot exist.

Proof: Consider two vectors X, Y lying on the unit hypercube. Let the "Hamming—Like distance" between them be 'd'. The inner product of X, Y is given by'

< X , Y > = { Number of components where X, Y agree }–{ Number of components where X, Y disagree }

= (N – d) - d = N – 2 d .

Thus for the two vectors X , Y to be orthogonal, it is necessary and sufficient that "N" is an even number. Thus, Hadamard matrices of odd order donot exist. **Q.E.D**

Corollary: If the dimension is odd, there are no orthogonal vectors lying on any one of the countably many hypercubes.

Thus, we are interested in determining whether Hadamard matrices of any arbitrary EVEN ORDER exist. A partial answer to this question is well known. Specifically, it is known that Hadamard matrices of order 2^k exist for all $k \geq 0$. The so called Sylvester construction is provided below:

$$H_1 = [1]$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}.$$

- **Motivation for Real Hadamard Transform**:

  It is clear that the rows of real Hadamard matrix can be interpreted as orthogonal basis functions on the interval [0,1] or more generally [0,T] for a finite real number T. These orthogonal basis functions are variously called as Walsh, Hadamard, Rademacher functions. As in the case of other transforms, these basis functions can be utilized to define and study a new transform. Specifically, real Hadamard transform decomposes an arbitrary input vector into superposition of Walsh functions. It is an example of generalized class of Fourier transforms. Specifically, it can be interpreted as as being built out of size-2 discrete Fourier transforms.

- **Details of Hadamard transform:**

  The real Hadamard transform is a real Hadamard matrix (normalized / scaled by a normalization factor). Specifically H_m is a 2^m x 2^m matrix, the Hadamard matrix (scaled by a normalization factor), that transforms 2^m real numbers x_n into 2^m real numbers X_k. The real Hadamard transform can be recursively defined in the following manner:

  The 1 x 1 Hadamard transform H_0 is defined as 1 i.e. $H_0 = 1$. Then H_m for m > 0 is defined in the following manner:

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & H_{m-1} \end{pmatrix},$$

  where the $\frac{1}{\sqrt{2}}$ is a normalization constant that is sometimes omitted.

- Hadamard transform, $\bar{v}$ of an N x 1 vector $\bar{u}$ is written as

  $\bar{v} = H \bar{u}$. (H is an N x N Hadamard matrix)

  and the inverse Hadamard Transform is given by

  $\bar{u} = H \bar{v}$.

- Hadamard transform is utilized in various applications such as image compression. It constitutes one among the various transforms utilized in, say image processing [4].

3. Novel Complex Hadamard Matrices: Complex Hadamard Transform:

In our research efforts [1],[2],[3] on complex valued neural networks, we encountered vectors whose components belong to the following set:

D = { 1+ j1 , 1 –j 1, -1 + j 1, -1 – j 1 }.

Definition : The finite collection of vectors whose componets belong to the set D are defined to lie on the "unit complex hypercube".

This definition is motivated by the definition of real valued symmetric unit hypercube). We now derive the conditions for two vectors on the complex hypercube to be unitary to each other. We provide necessary (not obvious condition) and sufficient condition.

- **Conditions on Unitary Vectors on Complex Hypercube:**

Let the two complex valued vectors lying on it be denoted by X, Y. They can be represented in the following manner:

X = A + j B, Y = C + j D , where

A, B, C, D lie on the real valued unit hypercube.

- **Definition:** X and Y unitary / orthogonal when

$X^* Y = 0$, $where$

X^* denotes the conjugate transpose of X.

Now we find the conditions for unitarity / orthogonality of X, Y. We need the following definitions. Let

d_1 $is\ the\ number\ of\ places\ where\ A, C\ differ$,
d_2 $is\ the\ number\ of\ places\ where\ B, D\ differ$,
d_3 $is\ the\ number\ of\ places\ where\ A, D\ differ$
d_4 $is\ the\ number\ of\ places\ where\ B, C\ differ$

Lemma 2: The vectors X and Y are unitary if and only if

$$d_1 = d_2 = d_3 = d_4 = \frac{N}{2}.$$

Proof: It is easy to see that

$$X^*Y = (A^T - j\, B^T)(C + j\, D)$$
$$= (A^T C + B^T D) + j\, (A^T D - B^T C) .$$

Now using the same idea as in Lemma 1, we have that

$$X^*Y = [(N - 2d_1 + N - 2\, d_2)] + j[(N - 2d_3) - (N - 2\, d_4)] .$$

Thus, we necessarily have that

$$X^*Y = [(2N- 2(d_1 + d_2)) + j\, 2(d_4 - d_3)] .$$

Thus, for X, Y to be unitary / orthogonal to one another, we must have that

N = $(d_1 + d_2)$ and $d_3 = d_4$.

Note: One sufficient condition for unitarity / orthogonality of X, Y is that the vectors A, B, C, D lying on the real valued unit hypercube are such that

$A^T C = 0$, $B^T D = 0$, $A^T D = 0$, $B^T C = 0$ i.e. Vector pairs (A,C) , (B,D), (A,D), (B,C) are orthogonal. This can happen if

$$d_1 = d_2 = d_3 = d_4 = \frac{N}{2}.$$

Now using the " identity swapping argument", it can be easily shown that the conditions

$$N = (d_1 + d_2) \text{ and } d_3 = d_4$$

necessarily require that

$$d_1 = d_2 = d_3 = d_4 = \frac{N}{2}.$$

Hence this condition is necessary as well as sufficient. Q. E.D

Suppose X, Y lie on the k-th order complex hypercube. Then, we have that the inner product of X, Y is expressed by the following formula.

$$X^* Y = 2K^2 [(N- (d_1 + d_2)) + j (d_4 - d_3)] .$$

Note: It should be noted that the above inner product is a complex number with even integer real and complex parts. It is a real number if and only if

$$d_3 = d_4 .$$

- **Remark:**

In the literature, there is an effort to study "complex Hadamard matrices" in which the elements are p^{th} roots of unity. But in view of the above discussion we are naturally led to the study of structured matrices whose elements are from the set

D = { +1+ j 1, +1 – j 1, -1 + j 1, -1 – j 1 }.

Equivalently, motivated by the definition of complex hypercube, we are naturally led to the study of "novel complex Hadamard matrices"whose rows and columns are corners of the complex hypercube. Formally, we have the following definition.

Definition: A Complex Hadamard matrix of order 'm', denoted by CH_m , is an m x m matrix whose elements belong to the set D such that

$$CH_m \, CH_m^* = 2m \; I_m$$

where I_m is the m x m identity matrix and CH_m^* is the conjugate transpose of CH_m. This definition is equivalent to saying that any two rows of H_m are unitary.

In view of the above definition and Lemma 4, we have the following interesting result

Lemma 3: The above novel Complex Hadamard matrices of odd order donot exist.

Proof: Follows directly from Lemma 2 Q.E.D.

- **Motivation for Complex Hadamard Transform**:

Consider a signal defined over finite support i.e. x[n], $0 \leq n \leq (N-1)$. It is well known that the Discrete Fourier Transform (DFT) of such a signal is given by

$$X(k) = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j\,2\,\pi\,n\,k}{N}} \quad for \quad 0 \leq k \leq (N-1).$$

In matrix-vector notation, the above equation can be rewritten as

$$\bar{X} = \overline{W}\,\bar{x} \text{ , where}$$

$$\bar{X} = [\, X(0)\; X(1) \ldots\ldots X(N-1)\,]^T$$

$$\bar{x} = [\, x(0)\; x(1) \ldots\ldots x(N-1)\,]^T \text{ and}$$

$\overline{W}$ is the DFT matrix whose elements are N^{th} roots of unity (i.e. complex numbers). Thus, unlike Real Hadamard Transform, Discrete Fourier Transform (DFT) as a transform is defined not only for real valued signals but also for complex valued signals. Thus, we are motivated to define and study complex Hadamard transform in the following discussion.

- **Complex Hadamard Transform**:

From the discussion above and the real valued Hadamard transform definition, it is clear that complex Hadamard transform is a Complex Hadamard matrix scaled by a normalization factor. i.e.

$$NCH_m = \frac{1}{\sqrt{2m}}\, CH_m .$$

Thus, it is clear that NCH_m is a unitary matrix. If such a matrix is constructed using the generalized Sylvester construction, it is also a Hermitian matrix. The order of such a matrix is a "power of 2".

Definition:

The Complex Hadamard transform of a complex valued vector $\bar{U}$ is obtained in the following manner:

$$\bar{V} = NCH_m\, \bar{U}.$$

Also, it is easy to see that (since NCH_m is unitary and Hermitian), the inverse complex Hadamard transform of $\bar{V}$ is given by

$$\bar{U} = NCH_m\, \bar{V}.$$

- In view of the above discussion, it is easy to see that the Complex Hadamard matrix NCH_m (i.e. also Complex Hadamard Transform) can be decomposed in the following manner

$$NCH_m = G + j\; H ,$$

where G and H are real Hadamard matrices (i.e. real Hadamard transforms).

- Consider two real input signals i.e. two real valued vectors $\bar{A}, \bar{B}$. Let us form a complex signal $\bar{U}$ in the following manner i.e.
  $$\bar{U} = \bar{A} + j\,\bar{B}$$
  Now let us compute the Complex Hadamard transform of the complex signal $\bar{U}$. It is easy to that
  $$\bar{V} = NCH_m\,\bar{U} = (\bar{G}\,\bar{A} - \bar{H}\,\bar{B}) + j\,(\bar{G}\,\bar{B} + \bar{H}\,\bar{A}).$$

  In a similar manner form another complex signal in the following manner:
  $$W = \bar{B} + j\,\bar{A}.$$
  The Complex Hadamard transform of the complex signal $\bar{W}$ is given by
  $$\bar{Z} = NCH_m\,\bar{W} = (\bar{G}\,\bar{B} - \bar{H}\,\bar{A}) + j\,(\bar{G}\,\bar{A} + \bar{H}\,\bar{B}).$$

  Thus, the complex valued vectors $\bar{V}, \bar{Z}$ can be decomposed in the following manner:
  $$\bar{V} = Real\ Part\ of\,(\bar{V}) + j\ Imaginary\ Part\ of\,(\bar{V}).$$
  $$\bar{Z} = Real\ Part\ of\,(\bar{Z}) + j\ Imaginary\ Part\ of\,(\bar{Z}).$$

  Using the above equations, we get the following results:

  (I) $\bar{G}\,\bar{A} = \frac{1}{2}\,(\ Real\ Part\ of\,(\bar{V}) + Imaginary\ Part\ of\,(\bar{Z})\,).$

  (II) $\bar{H}\,\bar{B} = \frac{1}{2}\,(\ Imaginary\ Part\ of\,(\bar{Z}) - Real\ Part\ of\,(\bar{V})\,).$

  (III) $\bar{G}\,\bar{B} = \frac{1}{2}\,(\ Real\ Part\ of\,(\bar{Z}) + Imaginary\ Part\ of\,(V)\,).$

  (IV) $\bar{H}\,\bar{A} = \frac{1}{2}\,(\ Imaginary\ Part\ of\,(\bar{V}) - Real\ Part\ of\,(\bar{Z})\,).$

- Thus, using the complex Hadamard transform of complex signals $\bar{U}, \bar{W}$ (formed from the real signals $\bar{A}, \bar{B}$), we are able to compute the real Hadamard transforms $\bar{G}\,\bar{A}$, $\bar{H}\,\bar{B}$, $\bar{G}\,\bar{B}$ and $\bar{H}\,\bar{A}$.
  The detailed properties and applications of complex Hadamard transform are discussed in [6, RaM]. It should be noted that real valued Hadamard transform finds many applications in research areas such as Digital Image Processing [4], [5].

- **Hadamard Transform : Threshold Decomposit**ion:

- It is well known that every finite dimensional linear operator has a matrix representation i.e. If "M" dimensional function of N variables $\bar{f}\,(X_1, X_2, \ldots X_N)$ satisfies the superposition property, then it can be represented in the following form:

  $\bar{f}\,(X_1, X_2, \ldots X_N) = G\,\bar{X}$, where G is an M x N matrix and $\bar{X} = [X_1, X_2, \ldots X_N]^T$.

  Using the principle of Threshold decomposition, the input $\bar{X}$ can be decomposed into binary signals. Suppose the components of vector $\bar{X}$ can assume say atmost L integer values. Then $\bar{X}$ can be decomposed in the following manner:

$\bar{X} = \bar{Y_1} + \bar{Y_2} + \cdots + \bar{Y_L}$, where $\bar{Y_l}$ are binary / Boolean vectors.

Thus $\bar{f}(X_1, X_2, \ldots X_N) = G\bar{X} = G\bar{Y_1} + G\bar{Y_2} + \cdots + G\bar{Y_L}$.

Hence such linear functions can be computed using Threshold decomposition. If G is a Hadamard matrix, then the computation on the right hand side involves only additions and subtractions.

Remark: It should be noted that finite impulse response linear filters, involve filtering operation on signal values in a finite length window. Thus such linear filters can be implemented using Threshold decomposition.

- **Cascaded Transforms**:

As per the above discussion, it is clear that the Discrete Fourier transform of a signal (real or complex valued) can be complex valued. The Complex Hadamard Transform of such a signal is then computed. The resulting signal is processed. After processing, inverse transforms are applied in the reverse order i.e. inverse Complex Hadamard transform is applied and then inverse DFT is computed. The resulting signal provides a processed approximation to the original signal. Effectively, we are computing cascaded transform of the original signal.

The above innovative idea of Complex Hadamard Transform leads to the following novel research directions.

- **Complex Walsh, Haar and Other Functions**:

The rows of complex Hadamard matrices (discussed above) are complex Walsh functions (could also be called complex Radamacher functions). Thus, in view of Lemma 2 on the unitarity of vectors (lying on complex hypercube), we are led to investigate orthogonality (unitarity) of two complex valued functions of the real variable.

- By definition, any pair of such functions f(t), g(t) are orthogonal / unitary on a finite domain, say [0,T] when their inner product is zero i.e.

$< f(t), g(t) > = \int_0^T f(t)\, g^*(t)\, dt = 0$, where $g^*(t)$ is the complex conjugate of g(t).

Let

$$f(t) = a(t) + j\, b(t) \quad \text{and} \quad g(t) = c(t) + j\, d(t).$$

Thus

$$< f(t), g(t) > = \int_0^T f(t)\, g^*(t)\, dt$$
$$= \int_0^T ([\, a(t)c^*(t) - b(t)\, d^*(t)\,]\; dt + j \int_0^T [\, b(t)c^*(t) + a(t)d^*(t)\,]\; dt.$$

Thus, the inner product of { f(t), g(t) } is zero if

< a(t), c(t) > = 0 and < b(t), d(t) > = 0 and < b(t), c(t) > = 0 and < a(t), d(t) > = 0.

These are sufficient conditions. In view of Lemma 2, they are also necessary conditions.

We now provide a method of synthesizing complex valued orthogonal functions { f(t), g(t) } of real variable 't'. For synthesizing such functions, let { a(t), b(t) } be real valued orthogonal functions of real variable 't'. Let us define

f(t) = a(t) + j a(t) and g(t) = b(t) + j b(t).

Based on the above discussion, it is easy to verify that { f(t), g(t) } are complex valued orthogonal functions of the real variable 't'.

In a similar manner, complex Haar functions are defined and studied [RaM].

Also, one can consider complex valued functions of complex variable i.e. J(z), K(z) where "z" is a complex variable. The range of functions { J(.), K(.) } is the set { 1+j1, 1-j1, -1+j1, -1-j1 }. Specifically, one can consider the domain of functions J(.), K(.) to be the set E i.e. E = { z : z = a + j b , where $a \in [0,1]\ and\ b \in [0,1]$ } Orthogonality of such functions is defined in a similar manner. Other interesting complex valued orthogonal functions are discussed in [RaM].

Note: It is well known that the concept of complex numbers is generalized to arrive at the concepts of quaternions and octanions. Thus, all the above results have natural generalization for quaternions and octanions.

4. Other Complex Transforms :

In the spirit of the above discussion related to Complex Hadamard Transform, we can define and study various other complex Transforms. For instance, let us consider the Haar transform [4], [5].

The basis functions of the Haar transform are the oldest and simplest known orthonormal wavelets. Haar transform can be expressed in the following matrix form:

$$T = H\ F\ H^T,$$

where F, for instance, is an image matrix, H is an N x N Haar transformation matrix and T is the resulting N x N Haar transform of F. The transpose is required because H is not symmetric. The Haar transform, H contains the well known Haar basis functions, $h_k(z)$. They are defined over the continuous closed interval $z \in [0,1]\ for\ k = 0,1,2,\dots,N-1,\ where\ N = 2^n$]. The rows of Haar Matrix, H can be generated in a well known recursive manner.

- Our idea is to define and utilize COMPLEX HAAR basis functions on the unit square [0,1] x [0,1]. The real and imaginary parts are the various possible real Haar basis functions.

- In a similar manner, complex Hough transform can be defined and studied. . More generally, we propose, “complex orthonormal wavelets” in the spirit of complex Haar basis functions. Also, complex bi-orthogonal wavelets are proposed. Detailed efforts are documented in [6, RaM].

- As in the case of real basis functions, complex basis functions such as complex Rademacher functions can be utilized as the basis functions to express a certain family of complex valued functions of real variable.

5. Conclusions:

In this research paper, a novel complex Hadamard matrix is defined based on the concept of complex hypercube. Utilizing the complex Hadamard matrix, complex Hadamard transform is defined and studied. In a natural way, other complex transforms are proposed. It is expected that the results in this paper lead to novel theoretical research directions and practical applications.

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