Quantum broadcasting of $3 \otimes 3$ negative partial transpose entangled states

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by

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To my parents,

my guiding lights in the dark

International Institute of Information Technology
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CERTIFICATE

It is certified that the work contained in this thesis, titled “Quantum Broadcasting of $3 \otimes 3$ Negative Partial Transpose Entangled States” by Sabuj Chattopadhyay, has been carried out under my supervision and is not submitted elsewhere for a degree.

_______________ Date

_____________________________________ Adviser: Dr. Indranil Chakrabarty

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Sabuj Chattopadhyay.

List of Publications

This thesis is largely comprised of content present in the following research paper:

Peer Reviewed Publication:

- R. Mundra, **S. Chattopadhyay**, I. Chakarabarty, N. Ganguly **Broadcasting of NPT Entanglement in Two Qutrit Systems**
  - Link: https://link.springer.com/article/10.1007/s12043-021-02098-w
  - **Pramana - Journal of Physics (Springer Publication)**: Accepted

Abstract

The utilization of quantum mechanical characteristics such as entanglement serves as a critical foundation in a multitude of quantum information processing tasks. With the objective of designing quantum networks, broadcasting - which is defined as the process of producing more quantum mechanical resources from a limited amount - presents an encouraging approach. One promising method of achieving broadcasting in quantum mechanical systems is through cloning operations. This thesis delves into the investigation of broadcasting of a class of quantum entangled states that go beyond $2 \otimes 2$ systems.

It is known that beyond $2 \otimes 2$ and $2 \otimes 3$ dimensional quantum systems, Peres-Horodecki criterion is no longer sufficient to detect separability of quantum states. This is because there exists entangled states with both positive and negative partial transpose (PPT and NPT). Further, it is also true that all PPT entangled states are bound entangled states. However, in the class of NPT states, there can exist bound entangled states as well as free entangled states. All free/useful/distillable entanglement is a part of the class of NPT entangled states. In this thesis, the question is asked that given an NPT entangled state in $3 \otimes 3$ dimensional system as a resource, how much entanglement can we broadcast so that resource still remains NPT.

The initial chapter offers a brief overview of the historical occurrences surrounding the evolution of Quantum mechanics. It is followed by a brief and succinct introduction to the necessary theory and mathematical tools required for this thesis.

In the next chapter a $3 \otimes 3$ system is chosen as a first step to understand broadcasting of NPT states in higher dimensional systems. In particular, we find out the range of broadcasting of NPT entanglement for Two parameter Class of States (TPCS) and Isotropic States (IS). Interestingly, as a derivative of this process we are also able to locate the existence of absolute PPT states (ABPPT) in $3 \otimes 3$ dimensional system. Here we implement the strategy of broadcasting through approximate cloning operations.

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Chapter 1

Preliminaries

> "The mind, once stretched by a new idea, never returns to its original dimensions"
>
> – *Ralph Waldo Emerson*

Max Planck's black body experiments conducted in 1900 marked the beginning of a long experimental history that eventually led to the modern formulation of quantum physics. The concept of "packets" of light or quantum was initially introduced in these experiments. This chapter presents a few of the significant historical milestones in quantum physics, described in Section 1.1, followed by a brief overview of the mathematical formalism. The reader is expected to have some basic understanding of linear algebra concepts. The postulates of quantum physics are then presented in mathematical form, along with relevant definitions like quantum state, linear operators, transformations, and measurements to help understand the postulates. The notion of quantum state representation as density operators is also explained, which is crucial for comprehending the presented work. Quantum systems have unique properties such as entanglement, superposition and nonlocality that can be utilized for information processing tasks, leading to the development of quantum information theory. The application of quantum information processing offers several advantages that far exceed the theoretical capabilities of classical information.

1.1 Historical Background of Quantum Theory

Quantum mechanics is widely considered to have originated with the resolution of the black body radiation paradox by Max Planck. With his model of black body, Planck was able to explain the full range of energy distribution of its emission spectrum by postulating the presence of countably infinite number of oscillators that could increase their energy only in small and minimal increments. These energy increments were proportional to the frequency of the emission radiation of the black body. The resulting formula was in agreement with that given by Wein's law as well as Rayleigh-Jeans law for small and large wavelengths respectively.

The efforts to reconcile the discrete energy posited by Planck with classical physics provided substantial theoretical challenges. The futility of such an enterprise came to light with the advent of Heisenberg's uncertainty principle in 1926. The principle states, (which was later verified from the experiments as well) that for certain pairs of canonically conjugate variables, it is impossible for those variables to simultaneously have values that can be determined with arbitrarily high precision and there is a necessary trade-off in precision of measurements of one variable with respect to the other. This principle was not an artifact of measurement errors but was a fundamental nature of reality itself.

Einstein made further developments in the theory by positing that it is not simply the electromagnetic oscillators of black body that have discrete properties, but electromagnetic radiation itself can come only in fixed packets or bundles of energy called "photons". This helped to explain the phenomenon of photoelectric effect for which Einstein was awarded the Nobel Prize in Physics in 1921. The particle nature of light was further corroborated by Compton Effect. A year later, Louis de Broglie postulated that associated with every particle is a plane harmonic wave described by the same kind of harmonic function which describes the electromagnetic wave, thus establishing the dual nature of particles and waves on theoretical grounds. This hypothesis was later confirmed experimentally by Davisson and Germer in their famous electron diffraction experiments.

Whether a quantum entity would show up the "wave" nature or the "particle" nature dependent on the type of the experiment that was being performed and what was being measured. Heisenberg and Schrodinger developed two formulations of quantum mechanics namely *matrix mechanics* and *wave mechanics* which were mathematically equivalent to each other.

Rolf Landauer, a physicist of German origin, was the first to propose that every piece of information must have a physical embodiment. Following this assertion, he delved into the thermodynamic implications of irreversible computations. Charles Bennett however demonstrated in the early part of the 1970s that these thermodynamic expenditures can be circumvented in the case of reversible computations. Initially, the primary focus of research in this discipline was on the constraints of classical information handling. However, this focus shifted following the revelation of nonlocal correlations within a qubit pair by John Bell in 1964. These correlations were of practical importance and were subsequently harnessed in the development of quantum cryptography by Bennett. Further research uncovered the potential of quantum information processing for enhancing the confidentiality of transmitted information, leading to the creation of several protocols for tasks related to quantum information processing, such as remote state preparation, entanglement swapping, teleportation, and superdense coding.

While there have been many experiments that have demonstrated various quintessential aspects of quantum phenomena, one of the classic ones was carried out in 1922 by Otto Stern and Walther Gerlach, which marked a turning point for empirical landscape of quantum mechanics. This experiment is described below.

Stern-Gerlach Experiment

Stern-Gerlach experiment is one of the seminal experiments in quantum theory which was responsible for uncovering a quintessential quantum property known as spin. The experiment also demonstrates quantum superposition and probabilistic measurement outcomes associated with a superposed state. The apparatus designed for the experiment constituted a spatially inhomogeneous magnetic field through which a beam of silver atoms was directed. Given the presence of an unpaired electron in the outer shell of each silver atom, the atom possesses a net magnetic moment attributable to the electron's intrinsic spin. As the atomic beam traverses the non-uniform magnetic field, it is subjected to a force, causing the beam to diverge.

According to principles of classical physics, the anticipated result would be a continuous distribution of deflections. This presumption is grounded in the idea that the magnetic moment can be orientated arbitrarily in space. However, in stark contrast to this classical prediction, the Stern-Gerlach experiment demonstrated a distinctly quantized outcome, with the beam bifurcating into two discrete components upon the interaction. This dichotomy corresponded to the two possible eigenvalues of the z-component of angular momentum, thus explicitly demonstrating the quantized nature of spin.

This result served as an initial, empirical affirmation of the concept of spin in quantum systems, a concept that had hitherto only been hypothesized. It strikingly underscored the marked divergence of quantum systems from classical predictions. Moreover, it provided an empirical basis for understanding that spin components, akin to other quantum observables, are quantized, taking values in integer or half-integer multiples of the reduced Planck's constant.

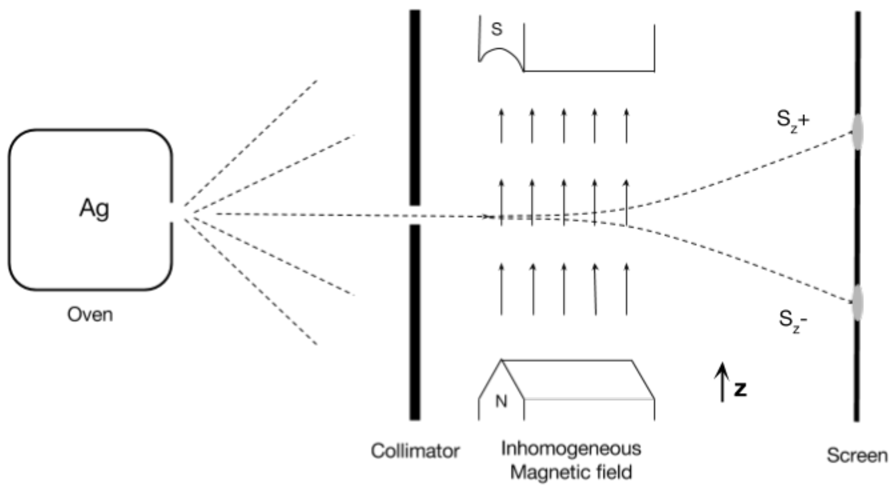

Figure 1.1: Schematic Depiction of the Stern-Gerlach Apparatus

Sequential Stern-Gerlach Experiments

Further examination of the Stern-Gerlach experiment offers profound insights into quantum measurement theory as well. Let's consider a silver atom emitted from a hot oven being fired into a Stern-Gerlach apparatus oriented in the z-direction (SG_z). The silver atom has a spin of $1/2$, and upon traversing the apparatus, it splits into two beams corresponding to the spin up and spin down states, denoted as SG_z+ and SG_z- respectively. This division of the atom's trajectory is an immediate consequence of space quantization, one of the cornerstones of quantum mechanics.

The first Stern-Gerlach apparatus (SG_z) is followed by another identical apparatus. Remarkably, if the atom is detected in the spin up state after the first SG_z and then routed into the second SG_z, it is again detected in the spin up state, post the second SG_z. This result may appear trivial, but it is essential in revealing the quantum nature of the spin.

If a third Stern-Gerlach device is introduced between the first two, oriented along the x-axis (SG_x), things become intriguing. Post the first SG_z apparatus, the silver atom enters SG_x and is split into two paths corresponding to the spin right (S_x+) and spin left (S_x-) states. If we choose to follow atoms in the spin right state and direct them into the second SG_z apparatus, we observe that these atoms split again into spin up and spin down states in the z-axis, contrary to the initial $SG_z - SG_z$ configuration.

This surprising result illustrates the principle of quantum superposition, where a quantum system (in this case, the silver atom's spin) can exist in a superposition of states. After the SG_x measurement, the spin state of the atom is a superposition of spin up and spin down states in the z-direction. When it enters the second SG_z apparatus, the superposed state 'collapses' due to the measurement, giving rise to spin up and spin down outcomes with equal probabilities. Moreover, it also demonstrates that the outcomes of measurements on quantum systems depend on the sequence of measurements made, a concept known as the "contextuality" of quantum mechanics.

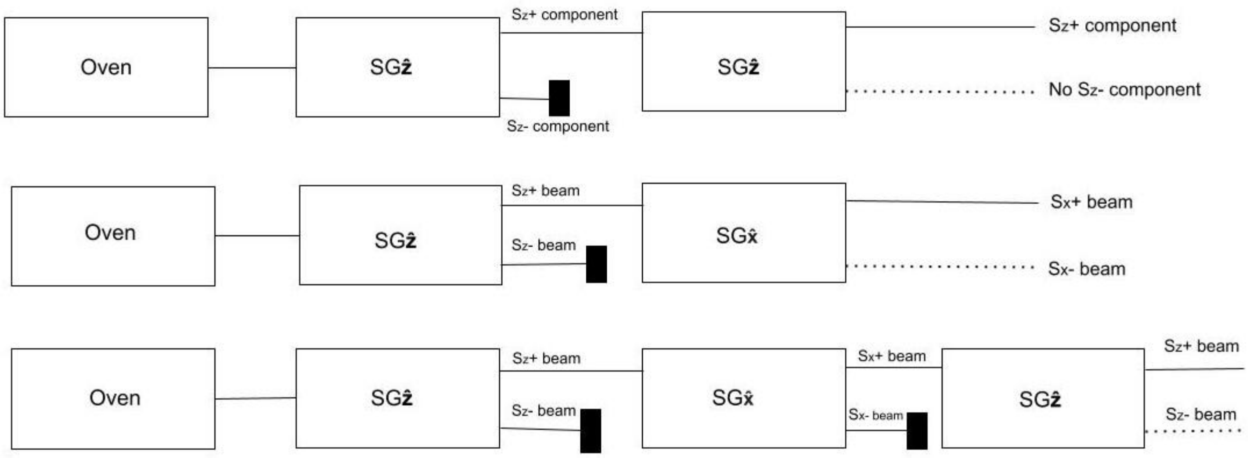

Figure 1.2: Sequential Stern-Gerlach Experiments

1.2 Mathematical Formalism

Quantum mechanics is formulated in the mathematical language of linear algebra and functional analysis. In the finite dimensional discrete quantum systems that this work deals with, the formalism of linear algebra is sufficient and the relevant concepts of the same required for full comprehension are as described below:

1.2.1 Hilbert Spaces

A Hilbert space $\mathcal{H}$, is a vector space with a sesquilinear inner product defined over a field of complex numbers $\mathbb{C}$, and two associated operations of addition and multiplication, such that they obey the following rules:

1. **Vector Addition**: For any elements $x, y, z \in H$, it follows

$$\begin{aligned} x + y &\in \mathcal{H} && \text{(Closure)} \\ x + y &= y + x && \text{(Commutative)} \\ (x + y) + z &= x + (y + z) && \text{(Associative)} \end{aligned} \tag{1.1}$$

There exists a vector $\mathbf{0}$, such that for any vector x in $\mathcal{H}$, $x + \mathbf{0} = \mathbf{0} + x = x$. There is also an additive inverse y for each x, such that $x+y = y+x = \mathbf{0}$.

2. **Scalar Multiplication**: For any two complex numbers $a, b \in \mathbb{C}$ and vectors $x, y \in \mathcal{H}$,

$$\begin{aligned} a.x &\in \mathcal{H} && \text{(Closure)} \\ a(b.x) &= (ab).x\,, \quad 0.x = \mathbf{0}\,, \quad 1.x = x && \text{(Identity)} \\ (a+b)x &= ax + bx\,, \quad a(x+y) = ax + ay && \text{(Distributive)} \end{aligned} \tag{1.2}$$

3. **Inner Product**: The *inner product* in a Hilbert Space is a function defined as $(*,*) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, mapping two vectors $x, y \in \mathcal{H}$ to a complex scalar in the underlying field with the following properties,

$$\begin{aligned} (x,x) \geq 0 \text{ and } (x,x) = 0 &\iff x = 0 && \text{(Positive-definite)} \\ (x,y) &= (y,x)^* && \text{(Conjugate Symmetry)} \\ (x,ay) &= a(x,y) \text{ and,} \\ (x, y_1 + y_2) &= (x,y_1) + (x,y_2) && \text{(Linear in second argument)} \end{aligned} \tag{1.3}$$

Two vectors x and y are orthogonal if and only if the inner product of them as defined above is $\mathbf{0}$, i.e $(x,y) = 0$. The inner product also allows us to calculate the *norm* of a vector. It is defined as $||x|| = +\sqrt{(x,x)}$. A unit vector is obtained by dividing the vector by its norm.

$$\hat{y} = \frac{y}{||y||}, \text{ such that } ||\hat{y}|| = 1. \tag{1.4}$$

A set comprising of vector elements $x_1, x_2 \ldots x_n$ in an n-dimensional vector space are said to be *linearly dependent*, if there exists $a_1, a_2 \ldots, a_n \in \mathbb{C}$ satisfying the equation $a_1x_1 + a_2x_2 \ldots a_nx_n = 0$ in which not all of them are zero. If the condition doesn't hold, the set of vectors are said to be linearly dependent. It's also the case that every linearly independent set of vectors which span the vector space forms a *basis*. For an n-dimensional vector space, the basis comprises of n linearly independent vectors. An *orthonormal* basis is defined by the condition that it satisfies $(x_i, x_j) = \delta_{ij}$ where $i, j = 1, 2 \ldots n$.

A Hilbert space with an uncountably infinite dimensional basis is also called as a *wave function* space as the set Hilbert space consists of square integrable functions (ψ) which are the *state vectors* in that case.

1.2.2 Linear Operators And Their Hermitian Adjoint

Transformations on vector space $\mathcal{H}$ are defined by linear operators which are functions of the form $A : H \to \Phi$ which map each vector $x \in \mathcal{H}$ to another vector $y \in \mathcal{H}$, written as $Ax = y$. The transforation is made linear by the imposition of following two conditions: i) $A(x+y) = Ax+Ay$, and ii) $A(ax) = a(Ay)$. In place of ii) above, if the condition was $A(a\varphi) = a^*(A\varphi)$, it would be a condition for anti-linearity. The operations of linear operators follow the following rules:

1. $(A+B)x = Ax + Bx$
2. $(aA)x = aAx$
3. $(AB)x = A(Bx)$

There is also an adjoint operator $A^\dagger$, defined for every operator A on $\mathcal{H}$, such that $(A^\dagger x, y) = (x, Ay)$ for every $x, y \in \mathcal{H}$. In a matrix representation of the operator A, the adjoint, $A^\dagger$ is obtained by transposing the matrix and then taking the complex conjugate of each element. These two operations are independent of the order in which they are performed. If $A^\dagger = A$, then the operator is called *self-adjoint* or *Hermitian*. In quantum mechanics for every observable of a system, there is an associated Hermitian operator.

If a linear operator preserves the magnitude of any vector on which it acts, then the operator is called *Unitary transformation*. Therefore the additional properties of a unitary operator U, are: $U^\dagger U = UU^\dagger = I$ or $U^\dagger = U^{-1}$.

Two linear operators A and B are said to commute if and only if the quantity $[A, B] = AB - BA = 0$. The operator $[A, B]$ is called a *commutator*.

1.2.3 Eigenvalues and Eigenvectors

Any operator A acting on a vector space transforms the vector on which it is applied into another vector in the same space. However, for certain special vectors, the operation results in the vector only being scaled by a certain scaler factor. All such non-zero vectors are known as the eigenvectors of the operator A, while the scaler factor by which the eigenvector is scaled on being acted on by A is called its eigenvalue. In other words if for a certain vector x, $Ax = \lambda x$ where λ is a scaler quantity, then x is the eigenvector of A while is the eigenvalue corresponding to that.

1.2.4 Dirac Notation

A notation to represent vectors in Hilbert spaces and operators on them called was introduced by Paul Dirac, in 1939 called the *Bra-Ket* notation. Any vector x in a Hilbert space is denoted in this notation by $|x\rangle$, which is called a *ket* vector or simply a ket. Corresponding to each ket

$|x\rangle$ in the Hilbert space there exists an element in *dual* space $\langle x|$, called the *bra* vector. The bra and the ket vectors are related by $\langle x|^{\dagger} = |x\rangle$ where the symbol "$\dagger$" denotes the adjoint operation discussed earlier. $\langle x|y\rangle$ denotes the inner product of the two vectors $|x\rangle$ and $|y\rangle$

1.2.5 Outer Product Notation

Operators in quantum mechanics can be stated in terms of *outer products* which can be denoted in Dirac notation. If we have two vectors $|x\rangle$ in $\mathcal{H}_1$ and $|y\rangle$ in $\mathcal{H}_2$, the outer product of the two vectors is then denoted by: $|x\rangle\langle y|$, which is a linear operator from the $\mathcal{H}_1$ to $\mathcal{H}_2$. The operation of the outer product is defined as:

$$\big(\,|x\rangle\langle y|\,\big)\,\big|y'\big\rangle = \big\langle y\big|y'\big\rangle\,|x\rangle\ . \tag{1.5}$$

An identity operator in the Hilbert space $\mathcal{H}$ can be represented in terms of the following summation of the outer products of the elements in orthonormal basis set $\{|i\rangle\}$

$$\sum_i |i\rangle\langle i| = I\,, \tag{1.6}$$

This is called the *Completeness relation.* A collection of orthonormal vectors are complete only if the completeness relation is followed. In such a case they also constitute an orthonormal basis set. If a vector $|\Psi\rangle \in \mathcal{H}$, then we can use the completeness relation above to write $|\Psi\rangle$ as:

$$|\Psi\rangle = I\,|\Psi\rangle = \sum_i |i\rangle\,\langle i|\Psi\rangle = \sum_i c_i\,|i\rangle \quad \text{where } c_i = \langle i|\Psi\rangle\,. \tag{1.7}$$

For a linear operator $\hat{O}$ in $\mathcal{H}$, applying the completeness relation gives us:

$$\begin{aligned} A = I\hat{O}I &= \left(\sum_i |i\rangle\langle i|\right)\hat{O}\left(\sum_j |j\rangle\langle j|\right) \\ &= \sum_{i,j} |i\rangle\langle j|\ \langle i|\hat{O}|j\rangle\ , \end{aligned} \tag{1.8}$$

The $[\hat{O}]_{ij} = \langle i|\hat{O}|j\rangle$ gives us the matrix elements of the operator $\hat{O}$ in the basis set $\{|i\rangle\}$, and the $\{|i\rangle\langle j|\}$ form the operator basis in the Hilbert space $\mathcal{H}$.

1.2.6 Tensor Products

If we consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, with dimensions $\dim(\mathcal{H}_1) = D_1$ and $\dim(\mathcal{H}_2) = D_2$ with vectors within it given by $|x_1\rangle \in \mathcal{H}_1$ and $|x_2\rangle \in \mathcal{H}_\in$, we can construct a tensor product from the given vectors as $|x_1\rangle \otimes |x_2\rangle$ which is also denoted as $|x_1\rangle\,|x_2\rangle$ such that the following properties hold true:

1. The tensor product is linear in both the arguments

$$(\alpha |x_1\rangle + \beta |y_1\rangle) \otimes |x_2\rangle = \alpha |x_1\rangle \otimes |x_2\rangle + \beta |y_1\rangle \otimes |x_2\rangle$$
$$|x_1\rangle \otimes (\alpha |x_2\rangle + \beta |y_2\rangle) = \alpha |x_1\rangle \otimes |x_2\rangle + \beta |x_1\rangle \otimes |y_2\rangle \tag{1.9}$$

2. The inner product of the vectors $|x_1\rangle |x_2\rangle$ and $|y_1\rangle |y_2\rangle$ is given by $\langle x_1|y_1\rangle \langle x_2|y_2\rangle$.

3. If we take the tensor products of all the vectors from the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, they form another Hilbert space, $\mathcal{H}_1 \otimes \mathcal{H}_2$, of dimension $D_1 D_2$. If $\{|i_1\rangle\}$ and $\{|j_2\rangle\}$ constitute the orthonormal basis sets for the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the tensor products of the $D_1 D_2$ pairs, $|i_1\rangle |j_2\rangle$, forms the orthonormal basis set for the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

4. If $\hat{O}_1$ and $\hat{O}_2$ are linear operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the tensor product $\hat{O}_1 \otimes \hat{O}_2$ is a linear operator on the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

1.3 Postulates of Quantum Mechanics

The mathematical apparatus of quantum mechanics is given by the following axiomatic postulates which describe the description of the state and its dynamics. [3, 5]

Postulate 1 The quantum mechanical state of a system is denoted by *wavefunction* $\Psi(x, t)$ which is a single-valued, continuous and square integrable function that encodes complete information about the system. It gives the probability amplitude, while the actual probability of locating the system within the coordinate range of x to $x + \mathrm{d}x$ is given by $\Psi(x, t)^* \Psi(x, t)\, \mathrm{d}x$.

Given that the probability of finding the system somewhere in the whole coordinate space is always 1, we can conclude that the following equation always holds:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \Psi(x, t)^* \Psi(x, t) = 1\,. \tag{1.10}$$

In a similar manner, in order to calculate the probability that the system lies in the coordinate interval $[a, b]$, the probability of finding the state in an infinitesimal interval x to $x + \mathrm{d}x$ is integrated over the interval $[a, b]$:

$$\int_{a}^{b} \mathrm{d}x\, \Psi(x, t)^* \Psi(x, t) \tag{1.11}$$

This procedure for calculating probability was given by the German physicist Max Born and therefore is referred to as the Born rule.

We can associate wavefunction $\Psi(x,t)$ with the abstract Hilbert space element (state vector) ket $|\Psi\rangle$ as $\Psi(x,t) = \langle x|\Psi\rangle$.

Postulate 2 There is a correspondence between the physical observables of a system and Hermitian operators which acts on the state vector. Hermitian operators have the mathematical property that all its eigenvalues are real and this set of real eigenvalues constitute the possible measurement outcomes of the physical observables.

Postulate 3 A state vector $|\Psi\rangle$ can be written in terms of coherent superposition of orthonormal eigenvectors $\{|\psi_i\rangle\}$ of some operator $\hat{O}$ which forms a complete basis set (i.e $\sum_i |\psi_i\rangle\langle\psi_i| = I$),

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle\,, \tag{1.12}$$

The expectation value of operator $\hat{O}$ with respect to state Ψ can be expanded in terms of its eigenvectors as,

$$\begin{aligned}\langle\hat{O}\rangle_\Psi = \langle\Psi|\hat{O}|\Psi\rangle &= \sum_{i,j} c_i^* c_j \langle\psi_i|\hat{O}|\psi_j\rangle \\ &= \sum_{i,j} c_i^* c_j o_j \langle\psi_i|\psi_j\rangle \\ &= \sum_{i,j} c_i^* c_j o_j \delta_{ij} = \sum_j |c_j|^2 o_j\,.\end{aligned} \tag{1.13}$$

An operator $\hat{O}$ is positive if all the eigenvalues of the operator are positive.

Postulate 4 For a normalized state vector $|\Psi\rangle$, the probability that the measurement outcome o_n of an operator $\hat{O}$ is obtained is given by $\mathcal{P}(o_n) = |\langle\psi_n|\Psi\rangle|^2$ where $|\psi_n\rangle$ is an eigenvector of the operator $\hat{O}$ with eigenvalue o_n (assuming non-degeneracy). If there is a degeneracy in eigenvalues, say g_n of the eigenvalue o_n then the probability of the outcome can be computed as $\mathcal{P}(o_n) = \sum_{i=1}^{g_n} |\langle\psi_n|\Psi\rangle|^2$.

Postulate 5 Once an observable $\hat{O}$ is measured on the state $|\Psi\rangle$ and an outcome o_n is realized, then the initial state undergoes a change. The new state immediately after measurement is given by the normalized projection of $|\Psi\rangle$ onto the eigenspace of o_n i.e,

$$\frac{P_n|\Psi\rangle}{\sqrt{\langle\Psi|P_n|\Psi\rangle}}, \qquad \text{where the projection } P_n = |\psi_n\rangle\langle\psi_n| \tag{1.14}$$

Postulate 6 The dynamic evolution of a quantum state $\Psi(t)$ is given by the *(Time Dependent) Schrödinger Equation*,

$$\imath\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle \tag{1.15}$$

where $\hat{H}(t)$ denotes the *Hamiltonian* operator. The Hamiltonian gives the total energy of the system.

1.3.1 A generalized two dimensional state in $\mathbb{C}^2$ represented as a point on a Bloch Sphere

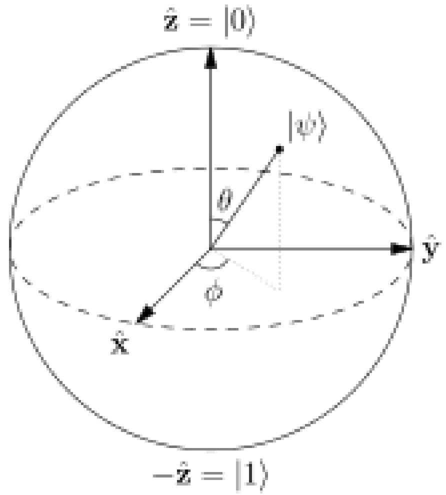

Figure 1.3: The Bloch sphere (3-dimensional unit sphere) representation of states in $\mathbb{C}^2$ state space. All the states are rays originating from the center of the sphere, with the pure states lying on the surface while mixed states make up the interior of the sphere [1].

A single qubit can be represented geometrically as a point on a Bloch sphere. Every point on the surface of the unit 3-dimensional sphere represents a pure state and the state can be characterized by the parameters $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi]$ which are the polar coordinates of the point on the sphere. Therefore, in general any pure qubit state can be written down in the form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{\iota\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \ , \tag{1.16}$$

where $|0\rangle$ and $|1\rangle$ are the basis states, and are represented by the poles of the sphere as shown in Fig.1.3 [1]. All possible quantum states fall on the mathematical rays which originates from the center of the Bloch sphere and which can be described by a vector $\vec{n}$ in $\mathbb{R}^3$. If the states are pure, then they lie on the surface of the sphere (where $|\hat{n}| = 1$). On the other hand, the mixed states lie inside the sphere where $|\vec{n}| < 1$. The vector $\vec{n} \equiv (n_x, n_y, n_z)$ in the generalized representation of the mixed state $\rho = \frac{1}{2}(I + \vec{n} \cdot \vec{\sigma})$ gives the coordinate of the state on the sphere, and is referred to as the Bloch vector. This representation can also be helpful as rotations of vectors corresponding to a particular quantum state can also be interpreted as unitary operations on the given state. Even though it is simple to visualize Bloch sphere for qubits, there is no straight forward way to visualize bloch spheres for quantum states in higher dimensions.

1.3.2 Ehrenfest Theorem

The Schrödinger equation (1.15) which was described in Postulate 6 above, is linear and homogeneous. This means that the superposition of different solutions to the equations is also a solution. However another property of relevance in quantum information is the that the probability structure itself is preserved.[3] Ehrensfest Theorem gives us a way to compute how the expectation value of observables evolves with time using the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = \frac{1}{\imath\hbar}\left\langle\left[\hat{O},\hat{H}(t)\right]\right\rangle + \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle \tag{1.17}$$

1.3.3 Measurements in Quantum Mechanics

For a given measurement, if m denotes the possible outcomes of the measurement, then we can denote a set of operators $\{M_m\}$ on the Hilbert space as general measurement class such that, if before measurement, the system is in the state $|\Psi\rangle$, then the probability with which the outcome m is obtained after the measurement is given by

$$p(m) = \langle\Psi|M_m^\dagger M_m|\Psi\rangle\,, \quad \text{and the state after measurement is given by,} \qquad \frac{M_m\,|\Psi\rangle}{\sqrt{p(m)}}. \tag{1.18}$$

All measurement operators must be completely related, i.e $\sum_m M_m^\dagger M_m = I$, (this implies conservation of probability), $\sum_m p(m) = 1$.Below,two special classes of measurements are discussed which are i) Projective Measurements and ii) Positive Operator Valued Measurements, also abbreviated as POVM [1]

Projective Measurements

If an observable $\hat{O}$ has a spectral decomposition,

$$\hat{O} = \sum_m o_m P_m,$$

then P_m is called the projector onto the eigenspace of the operator $\hat{O}$, with the eigenvalue o_m. These projectors are the measurements operators and the probability of outcome m and the state after measurement and the state after the measurement respectively are given by,

$$p(m) = \langle\Psi|P_m|\Psi\rangle \quad \text{and,} \qquad |\Psi\rangle \xrightarrow{\hat{O}} \frac{P_m\,|\Psi\rangle}{\sqrt{p(m)}} \tag{1.19}$$

If the operators M_m used in general measurements are hermitian or orthogonal projectors (along with obeying the completeness relation), then they can be treated as projective measurements. When the set $\{|m\rangle\}$ constitute an orthonormal basis, the corresponding projector for the vector $|m\rangle$ is given by $P_m = |m\rangle\langle m|$. In projective measurements, if the outcomes of first measurement is o_m for a particular observable $\hat{O}$, then repeating the same measurement of the given observable will yield same outcomes.

1.3.4 Density Operator

Quantum mechanics when expressed in terms of density operators is mathematically equivalent to the state vector formalism. In real world experiments conducted in laboratory conditions however, we seldom possess the complete information about the state and have no choice but to work with impure states. In these situations, the equations of quantum mechanics expressed in terms of density operators have several advantages over the state vector formalism. This is because while the given state may never be perfectly known, it is still possible for the experimenter to be able to describe the system being in a certain state $|\psi_k\rangle$, with probability p_k. It is thus possible to describe the state in terms of an ensemble or a mixture of pure states $\{|\psi_k\rangle\}$ which have respective probabilities $\{p_k\}$ such that the condition $\sum_k p_k = 1$ is satisfied. This probabilistic ensemble is also known as *mixed state*. This is in contrast to *pure state* when the state is completely known.

The probabilities in this case arise due to two distinct factors. The first is due to the system being described as a statistical mixture and therefore behaving as per the principles of classical probability. The second factor comes into play when measurement is done on the system and the intrinsic quantum mechanical probabilities come to surface. Hence the statistical mixture is in general distinct from and should not be confused with the pure superposition state $|\Psi\rangle = \sum_k c_k |\psi_k\rangle$. The density operator in general contains interference terms which makes describing a statistical mixture by an average state vector an impossible task. [3]

Density operators for pure states

If we examine a pure state, $|\Psi\rangle = \sum_k c_k |u_k\rangle$, where the basis $\{|u_k\rangle\}$ is orthonormal and $\sum_k |c_k|^2 = 1$ (the normalization condition), then the density operator is created by simply forming the projector $\rho = |\Psi\rangle\langle\Psi|$. This operator is a matrix and the matrix elements are,

$$[\rho]_{mn} = \langle u_m|\rho|u_n\rangle = \langle u_m|\Psi\rangle \langle\Psi|u_n\rangle = c_n^* c_m \tag{1.20}$$

The condition for normalization for the state vector,which is $\sum_k |c_k|^2 = 1$, can be expressed in terms of density operator as,

$$\sum_k |c_k|^2 = \sum_k [\rho]_{kk} = \mathrm{Tr}(\rho) = 1 \tag{1.21}$$

For any given observable $\hat{O}$, its expectation value is given by

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle = \sum_{n,m} c_n^* c_m \langle u_n | \hat{O} | u_m \rangle = \sum_{n,m} c_n^* c_m [\hat{O}]_{nm} \tag{1.22}$$

The elements present in the matrix of the observable $\hat{O}$ are denoted as $[\hat{O}]_{nm}$. Combining equations (1.20) and (1.22), the expectation value of $\hat{O}$ can be re-written as follows,

$$\begin{aligned} \langle \hat{O} \rangle &= \sum_{n,m} \langle u_m | \rho | u_n \rangle \langle u_n | \hat{O} | u_m \rangle \\ &= \sum_m \langle u_m | \rho \hat{O} | u_m \rangle = \mathrm{Tr}\left(\rho \hat{O} \right) \qquad \left(\text{since } \textstyle\sum_n |u_n\rangle\langle u_n| = I\right) \end{aligned} \tag{1.23}$$

To evaluate how the density operator evolves with time $|\Psi\rangle$ we begin with the Shrödinger equation given by,

$$\iota\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi\rangle = \hat{H} |\Psi\rangle . \tag{1.24}$$

Let's evaluate the time derivative of the density operator ρ

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} (|\Psi\rangle\langle\Psi|) = \left(\frac{\mathrm{d}}{\mathrm{d}t} |\Psi\rangle \right) \langle\Psi| + |\Psi\rangle \left(\frac{\mathrm{d}}{\mathrm{d}t} \langle\Psi| \right) \\ &= \frac{1}{\iota\hbar} \hat{H} |\Psi\rangle\langle\Psi| + \frac{1}{-\iota\hbar} |\Psi\rangle\langle\Psi| \hat{H} \qquad (\text{ From (1.24) }) \\ \iota\hbar \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \left[\hat{H}, \rho \right] \end{aligned} \tag{1.25}$$

which gives the Schrödinger evolution in the density operator formalism.

The definition of the density operator for pure states can yield certain useful properties, such as

- The density operator $\rho = |\Psi\rangle\langle\Psi|$ represents any two state vectors, which differ only in phase, such as $|\Psi\rangle$ and $e^{\iota\theta} |\Psi\rangle$, as the same state.
- The hermiticity is satisfied by the operator, i.e $\rho^\dagger = \rho$.
- Since ρ is a projector, it follows, $\rho^2 = \rho$.
- $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$.

The last two properties are only valid for pure states.

Density operators for mixed states

If we have a collection of pure states $|\psi_k\rangle$, then the density operator of their classical mixture, i.e the mixed state form can be defined as:

$$\rho = \sum_k p_k \rho_k = \sum_k p_k \left|\psi_k\middle\rangle\middle\langle\psi_k\right| , \tag{1.26}$$

In the equation, the coefficients $\{p_k\}$ are the probabilities that are assigned to individual states and they follow the ordinary laws of classical probability, which is each $0 \leq p_i \leq 1$ and the sum of probabilities is unity, $\sum_k p_k = 1$. For a measurement done on a mixed state, the probability of a certain outcome getting realized is the same as the sum of probabilities of that same outcome getting realized in all the individual pure states comprising the mixed state, each sum being weighted by the weight of the individual pure state in the mixed state. If a measurement $\hat{O} = \sum_n o_n P_n$ is done on a pure state $|\psi_k\rangle$, then the probability of outcome o_n getting realized is given by $\mathrm{Tr}(\rho_k P_n)$. Therefore the probability of the same outcome of the same measurement when done on the mixed state in terms of its density operator is:

$$\begin{aligned}\mathcal{P}(o_n) &= \sum_k p_k \mathcal{P}_k(o_n) = \sum_k p_k \,\mathrm{Tr}(\rho_k P_n) \\ &= \mathrm{Tr}\left[\left(\sum_k p_k \rho_k\right) P_n\right] \\ &= \mathrm{Tr}[\rho P_n] \end{aligned} \tag{1.27}$$

As with any density operator, the density operator for mixed states is Hermitian as well and its trace is 1.

$$\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \,\mathrm{Tr}(\rho_k) = \sum_k p_k = 1. \tag{1.28}$$

For any observable, $\hat{O}$, it's expectation value is given by $\langle \hat{O} \rangle = \mathrm{Tr}(\rho \hat{O})$, while the equation governing the time evolution of density operators for mixed states is the same as that of pure states, i.e

$$\imath\hbar \frac{\mathrm{d}\rho}{\mathrm{d}t} = \left[\hat{H}, \rho\right] \tag{1.29}$$

The positivity property holds for ρ in this case as well.

There are couple of properties that distinguish density operators for mixed states vis-à-vis pure states. If the states are mixed, then $\rho^2 \neq \rho$. Therefore $\mathrm{Tr}(\rho^2) < 1$

Density operator for a general state in $\mathbb{C}^2$

We have already seen from the postulates of quantum mechanics that every quantum state is associated with a Hilbert space. The Hilbert space dimensions depend on the system in question. In the case of two dimensional systems, the associated Hilbert space is denoted as $\mathbb{C}^2$. A density operator that acts on $\mathbb{C}^2$ would be a 2×2 matrix. This operator would have basis states as well, and we can take the set of outer products $|0\rangle\langle 0|$, $|0\rangle\langle 1|$, $|1\rangle\langle 0|$ and $|1\rangle\langle 1|$ as the basis. So any density operator can thus be represented as a linear combination of these states. However, there can be other bases also. One of the most useful ones in practice is to take I, σ_x, σ_y and σ_z as the basis which are the Pauli spin matrices and the identity matrix. This choice of basis enables us to visually render the state in an intuitive form in Bloch sphere representation. In this representation, operators can be visualized as rotations of the state vector on the sphere.

Any density operator in $\mathbb{C}^2$ can therefore be written in terms of the basis states of Pauli matrices and identity operator: $\rho = \alpha_0 I + \vec{\alpha} \cdot \vec{\sigma}$. a_0 is a complex number and $\vec{\alpha}$ can be thought of as a vector in three dimensional complex vector space $\mathbb{C}^3$ with components $\alpha_1, \alpha_2, \alpha_3$. $\vec{\sigma}$ is the vector comprising of the three Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$. Therefore $\vec{\alpha} \cdot \vec{\sigma} = \alpha_1\sigma_x + \alpha_2\sigma_y + \alpha_3\sigma_z$.

As with any density operator, ρ in $\mathbb{C}^2$ must also satisfy certain conditions which have been elaborated in previous sections. From the hermiticity condition, we get $\alpha_0 \in \mathbb{R}$ and $\vec{\alpha} \in \mathbb{R}^3$. Since the trace of a density operator is unity, we have $\mathrm{Tr}(\rho) = \alpha_0 \, \mathrm{Tr}(I) = 2\alpha_0$, which therefore implies $\alpha_0 = 1/2$. From the positivity condition of ρ, we get: $|\vec{\alpha}| \leq \alpha_0$ which also means $|\vec{\alpha}| \leq 1/2$

Once these parameter values are inserted into the generic expression of a density operator, it can be written as:

$$\rho = \frac{1}{2} I + \vec{\alpha} \cdot \vec{\sigma} \tag{1.30}$$

If we denote $2\vec{\alpha}$ by $\vec{n}$, then it follows $|\vec{n}| \leq 1$, therefore an arbitrary state in $\mathbb{C}^2$ can be written as:

$$\rho = \frac{1}{2} \left(I + \vec{n} \cdot \vec{\sigma} \right) , \tag{1.31}$$

This is for a single qubit. If we have a two qubit system, the general expression for its density matrix is given by:

$$\rho_{AB} = \frac{1}{4} \left(I \otimes I + I \otimes \vec{r} \cdot \vec{\sigma} + \vec{s} \cdot \vec{\sigma} \otimes I + \sum_{i,j=1}^{3} [T_{AB}]_{ij} \; \sigma_i \otimes \sigma_j \right) \tag{1.32}$$

Here $\vec{r}, \vec{s}$ belong to the three dimensional real space $\mathbb{R}^3$. T_{AB} is referred to as correlation matrix and and its matrix elements ($[T_{AB}]_{ij}$) are given by: $\text{Tr}(\rho_{AB}\sigma_i \otimes \sigma_j$

A general state comprising of n-qubits can be written as:

$$\rho_{AB} = \frac{1}{4}\left(I \otimes I + I \otimes \vec{r} \cdot \vec{\sigma} + \vec{s} \cdot \vec{\sigma} \otimes I + \sum_{i,j=1}^{3} [T_{AB}]_{ij}\ \sigma_i \otimes \sigma_j\right) \tag{1.33}$$

where $\vec{r}, \vec{s} \in \mathbb{R}^3$ and T_{AB} is called the correlation matrix with real matrix elements $[T_{AB}]_{ij} = \text{Tr}(\rho_{AB}\sigma_i \otimes \sigma_j)$. Extending this design to n-qubits, an arbitrary state ρ can be written as:

$$\rho = \sum_{i_1 \ldots i_n = 0}^{3} [T_n]_{i_1 \ldots i_n} (\sigma_{i_1} \otimes \sigma_{i_2} \cdots \otimes \sigma_{i_n}), \tag{1.34}$$

The identity operator is represented by σ_0 and σ_{i_k} are Pauli spin operators which acts on the k^{th} qubit. $[T_n]_{i_1 \ldots i_n}$ represents correlation tensor of dimensions $3^{\otimes n}$.

The quantum mechanical postulates can thus be re-written in terms of density operator formalism and mathematically they are equivalent to the pure state formalism described in the beginning of this section. The reformulated postulates are described below:

Postulate 1 Every distinct physical system is affiliated with a complex vector space possessing an inner product, otherwise known as a Hilbert space which serves as the system's state space. The system in its entirety is described via its density operator, symbolized as ρ. This is a positive operator with a trace of one and operates within the system's state space. If a quantum system exists in a state designated as ρ_i with a probability denoted as p_i, then the system's density operator is given by $\sum_i p_i \rho_i$.

Postulate 2 The progression of a closed quantum system with time is delineated by a transformation that maintains unitarity. Specifically, the state ρ at an initial time point t_1 connects to the state ρ' at a later time t_2 through a unitary operator U, which is solely dependent on the times t_1 and t_2. Hence, the transformation is expressed as $\rho' = U\rho U^\dagger$

Postulate 3 Measurements in quantum mechanics are characterized by an ensemble $\{M_m\}$ of operators which function on the given quantum system's state space. The subscript m denotes the potential outcomes of the measurement. Assuming the quantum system's state is ρ just before measurement, the likelihood of observing the the measurement outcome m is determined by:

$$p(m) = tr(M_m^\dagger M_m \rho) \tag{1.35}$$

Post measurement, the system's state is given by:

$$\frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)} \tag{1.36}$$

The set of operators $\{M_m\}$ are complete, i.e $\sum_m M_m^\dagger M_m = I$

Postulate 4 The state space corresponding to a composite physical system equates to the tensor product of the state spaces belonging to the individual components of that system. Furthermore, consider we have multiple systems identified numerically from 1 up to n, with each system i being readied in the state ρ_i. In this case, the collective state of the entire system is represented as $\rho_1 \otimes \rho_2 \otimes ...\rho_n$.

Postulate 5 The Schrödinger Equation formulated in terms of density operator is given by:

$$\iota\hbar \frac{\mathrm{d}\rho}{\mathrm{d}t} = \left[\hat{H}, \rho\right] \tag{1.37}$$

Partial Trace

It is possible to create a large quantum system by combining numerous smaller quantum systems. In such composites, the larger system has a vector space which is the tensor product of the vector spaces of the two or more separate systems. For instance, when two systems, A and B, with corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are combined, then, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space of the larger composite system. The density operator, which is ρ_{AB}, acts linearly on $\mathcal{H}$. The reduced density operator ρ_X for the subsystem $\mathcal{H}_X$ with $X = \{A, B\}$ can be obtained by performing a partial trace on the individual system not being considered. This helps us predict measurements on the subsystems. The matrix elements for the reduced density operator ρ_A can be found using the equation:

$$\left\langle n_A \middle| \rho_A \middle| n'_A \right\rangle = \sum_{\{|m_B\rangle\}} \langle n_A| \langle m_B| \rho_{AB} \left|n'_A\right\rangle |m_B\rangle \tag{1.38}$$

where, $|n_A\rangle$ and $|m_B\rangle$ form respectively the bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ and the summation is taken over all $|m_B\rangle$.

Similar calculation can give us the density operator in its reduced form for $\mathcal{H}_B$ and its matrix elements are denoted by $\langle m_B|\rho_B|m'_B\rangle$. It can be easily shown that $\mathrm{Tr}(\rho) = \mathrm{Tr}_A(\mathrm{Tr}_B(\rho)) = \mathrm{Tr}_B(\mathrm{Tr}_A(\rho))$. The partial traces of individual systems is one, i.e $\mathrm{Tr}(\rho_X) = 1$

Schmidt Decomposition

If there is a composite system AB, represented by the pure state vector $|\psi\rangle$, then this state can be written in the form of:

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \ , \tag{1.39}$$

Here the tensor product states correspond to orthonormal Schmidt basis, $|i_A\rangle$ and $|i_B\rangle$, for the quantum systems A and B respectively. These tensor product states have non-negative real coefficients λ_i, known as the Schmidt coefficients. The sum of the squares of the Schmidt coefficients is equal to 1 ($\sum_i \lambda_i^2 = 1$). A significant outcome of this decomposition is that the reduced density operators can be expressed as the sum of the squares of the Schmidt coefficients times the outer product of the corresponding Schmidt bases for subsystems A and B respectively, which is to say that $\rho_A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$ and $\rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$. It's clear that both of these reduced density matrices have the same eigenvalues. The Schmidt number, which refers to the number of non-zero Schmidt coefficients, is a crucial factor in determining whether a state is separable or entangled. A state is said to be in product form or separable if the Schmidt number is equal to 1, whereas an entangled state has Schmidt number greater than one.[1]

Purification

Purification is a theoretical construction that is inverse of the Schmidt decomposition. It refers to a mathematical process which augments a mixed state with an additional state in order to obtain a pure state. If we consider a mixed state to be ρ_A, then we can consider augmenting it with a reference state R so that we form a pure state $|AR\rangle$ while $\rho_A = \mathrm{Tr}_R(|AR\rangle\langle AR|)$. Purification is nothing but constructing such pure states. If we assume that the system A can be decomposed orthonormally as $\rho_A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$, then the definition of $|AR\rangle$ (with ortho-normal basis set $|i_R\rangle$) is:

$$|AR\rangle = \sum_i \lambda_i |i_A\rangle |i_R\rangle \ . \tag{1.40}$$

If we take the partial trace of this state over the additional reference state, then we get the density operator of our initial state A. Therefore $|AR\rangle$ is a purified form of ρ_A. [1]

1.4 Quantum Entanglement

Entanglement is one of the most quintessential quantum phenomena. This quantum mechanical property is what enables the quantum systems to act as a resource in several information processing tasks starting from teleportation to cryptography to superdense coding. Any quantum state that is not separable is entangled. A state is separable if it can be written in terms of tensor product of distinct individual states, that is:

$$\rho^{AB} = \sum_i p_i \left(\rho_i^A \otimes \rho_i^B\right) \ , \tag{1.41}$$

The usual probability rules apply on p_i ($p_i > 0$ and $\sum_i p_i = 1$)

Checking whether a state is separable is one of the ways of detecting entanglement but it's not trivial. In particular, detecting separability of mixed states is particularly challenging. Entanglement can also be determined by the Schmidt number of the quantum state. However for the special cases of quantum states in Hilbert spaces of dimensions $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$, a necessary and sufficient condition to detect entanglement was given by Asher Peres and Horodecki. This criteria states that if the partial transpose of the joint density matrix of the composite state is also positive, then the state is separable i.e $(\rho_{mp,nq})^{T_2} = \rho_{mq,np}$ should have eigenvalues which are all non negative. If there are more than two subsystems in a given quantum state, then the state can be fully separable or partially separable depending on whether or not the system can be written in terms of tensor products of all the subsystems comprising the composite state.

1.5 PPT/Bound Entanglement and NPT Entanglement

For a composite state ρ_{ab} defined on a Hilbert Space $H_A \otimes H_B$, if the partial transpose of the state is postitive semi-definite, i.e all its eigenvalues are non-negative, then the state is said to have a positive partial transpose (PPT). All separable states, that is those states that can be represented as the convex combination of product states, are PPT. However, the converse is not true. There can exist PPT states that are not separable. These states form a class called "Bound Entanglement States". They are so called because such states cannot be "freed" or distilled into simple smaller entangled units using local operations and classical communication.

On the other hand, if the partial transpose of the state ρ_{ab} possesses at least one negative eigenvalue, then such states are called NPT states. All NPT states are invariably entangled. Unlike PPT entangled states, many NPT entangled states are "distillable", i.e it is possible to extract pure state entanglement from them, which can be used for various information processing tasks. The NPT entangled states hence elucidate the exploitable, resource-like character of quantum entanglement, contrasting with the intriguing, yet less utilizable, bound entangled states.

1.6 No-Go Theorems

No-go theorems form a significant cornerstone in quantum mechanics. These mathematical results articulate boundaries within the structure of the theory, explicitly negating certain presumptions or expectations which may otherwise seem appealing or intuitive. By determining what quantum mechanics does not allow, no-go theorems underscore some of the most profound and counter-intuitive features of the theory.

Bell's theorem, named after physicist John S. Bell, is one such impactful no-go theorem. Articulated in 1964, this theorem elucidates a fundamental incompatibility between the quantum mechanical predictions for measurements on entangled particles and the predictions of a class of theories known as local hidden variable theories. Bell's theorem illustrates that quantum correlations surpass any limit set by local hidden variable theories, thus excluding these theories as possible explanations or extensions of quantum mechanics.

Another influential no-go theorem is the Kochen-Specker theorem. This theorem dictates that it is not possible to consistently assign pre-existing, context-independent values to all quantum observables. In other words, it negates the existence of non-contextual hidden variable theories. This highlights the central role of the measurement context in quantum mechanics, reinforcing the inherently contextual nature of quantum properties.

No-cloning and no-deleting theorems also fall within the purview of no-go theorems. These state the impossibility of creating an identical copy of an arbitrary unknown quantum state, and the impossibility of deleting an arbitrary unknown quantum state, respectively. Both theorems arise due to the linearity and unitarity of quantum mechanics and have significant theoretical as well as practical implications in quantum information theory and quantum computing.

The no-broadcasting theorem is a generalization of no-cloning theorem and is therefore another no go theorem. The theorem states that given a set of states, there does not exist a transformation (quantum operation) such that it can replicate all states in the set identically. The no broadcasting theorem is described in much greater detail in Chapter 2.

1.7 Conclusion

This chapter laid down some of the mathematical and theoretical concepts that are central to understanding quantum mechanics. The next chapter delves into the notion of quantum broadcasting and how broadcasting procedures can be used to distribute key quantum resources such an entanglement and to generate important classes of quantum states. It will also discuss the research done on broadcasting till now and the key new results in broadcasting that are presented as a part of this thesis.

Chapter 2

Broadcasting Of Quantum Correlations

> “Quantum correlations are a manifestation of the truly non-classical behavior of the quantum world.”
>
> – *Anton Zeilinger*

One of the most crucial limitations in quantum mechanics is the impossibility of duplicating an unknown quantum state, popularly referred to as the "No Cloning Theorem."[15] Formally, this theorem states that no operation can map two distinct quantum states, which are non orthogonal, $\{\psi, \phi\}$ to the quantum states $\{\psi \otimes \psi, \phi \otimes \phi\}$. One of the key assumptions of quantum cryptography is no-cloning theorem and the validity of this theorem was the underlying assumption for the first cryptographic key generation protocol by Bennett and Brassard[11]. Even though it is impossible to perfectly clone an unknown quantum state, there is always a possibility to do approximate cloning of such states[46]. Additionally, probabilistic cloning machines were created to clone an arbitrary quantum state with non zero probability, if not perfectly.[31]

Buzek and Hillery’s 1996 proposal of an estimated quantum cloning machine guarantees that the quality of copies remains constant regardless of the initial input state. This type of machine is commonly known as a universal quantum cloning machine (UQCM)[46], and subsequent research has proven this machine to be optimal as well[29][18]. Additional quantum cloning machines have since been developed, including state-dependent quantum cloning machines, as well as probabilistic quantum cloning machines[26].

The chief resource in any quantum computational or information processing task is quantum entanglement[48], which makes it highly critical in phenomena such as teleportation[9], super dense coding[35] as well as cryptography[33][34]. The unique and salient characteristic of quantum information processing is that the information is stored or encoded in the in the nonlocal correlations between two particles which maybe separated spatially by large (even spacelike) distances. The value of a two particle entangled state as a resource lies in how pure its entanglement is. However, entanglement can be vulnerable to environmental noise, leading to decoherence and entanglement degradation - the infamous "quantum noise". Quantum entanglement distillation is an operation that extracts a smaller number of higher fidelity entangled states from a larger set of less reliable ones, effectively combating this quantum noise and enabling robust quantum operations[58]. This operation makes it possible to locally compress quantum information. The possibility of local compression naturally raises the question of the possibility to do the reverse, i.e whether it is possible to decompress the entanglement. A number of researchers have addressed this question under the umbrella topic of "broadcasting of entanglement"[46][50][59].

Broadcasting can be defined as operations that locally copy the nonlocal quantum correlations. From an orignal entanglement pair, two lesser entangled pair can be generated using local as well as non-local operations. It is trivial to broadcast classical information. However in quantum mechanics, this is not always possible. For instance, it is impossible to broadcast non commuting mixed states (as it was showed first by Barnum et al)[20]. Subsequently, with respect to multipartite states, a no-local-braoadcasting theorem was given by Marco Piani et al. in 2008[36]. Once we have these two theorems, a natural problem would be to figure out the relationship between the two. The two theorems were shown to be mathematically equivalent by Shunlong Luo in 2010 through his no unilocal broadcasting theorem[21]. A holistic picture of no-broadcasting emerges from the combination of the above three theorems[22].

By "perfect broadcasting", it is meant local broadcasting of a state ρ^{ab} - in a manner that there exist operations $\Omega_1 : S(\mathbb{H}^a) \rightarrow S(\mathbb{H}^{a1} \otimes \mathbb{H}^{a_2})$ and $\Omega_2 : S(\mathbb{H}^b) \rightarrow S(\mathbb{H}^{b1} \otimes \mathbb{H}^{b_2})$ such that the von Neumann mutual information of the states ρ^{ab}, $\rho^{a_1 b_1}$ and $\rho^{a_2 b_2}$ are identical. The states $\rho^{a_1 b_1}$ and $\rho^{a_2 b_2}$ can be determined by taking the partial trace of the state $\rho^{a_1 a_2 b_1 b_2}$ with respect to the appropriate bases[36]. It has been shown that local or unilocal broadcasting is possible iff the states are having classical or classical-quantum correlations in the respective cases[36][21][16].

Till now, our discussion centered around perfect broadcasting of general states. However broadcasting an entangled state entails creating more number of lesser entangled pairs from a more entangled pair and the von-Neumann information content in the states $\rho^{a_1 b_1}$ and $\rho^{a_2 b_2}$ is less than the initial state ρ^{ab}. This can be achieved by doing local cloning operations on each particle of the given entangled pair or by globally cloning the entire pair[46][50]. It was shown by Buzek et al. that even though it is not possible to perfectly broadcast an entangled state, it is possible to partially decompress an entangled state through local operations - i.e from a given highly entangled pair, produce two lesser entangled pairs[46]. If the states are pure, broadcasting and cloning are equivalent.

For broadcasting to be successful it is essential that nonlocal output states are inseparable while the local output states are separable. Universal cloners which have a fidelity greater than $\frac{1}{2}\left(1+\sqrt{\frac{1}{3}}\right)$ are the only ones suitable for the purpose of broadcasting because only these cloners make it possible for the non-local output states to become inseparable (as shown by Bandyopadhyay et al)[50]. The optimal broadcasting happens only when optimal cloners are used to do the local operations. It is also not possible to do the broadcasting into three or more entangled pairs using only local cloning operations.

2.1 No-Broadcasting Theorem

This section will give a brief description of the No-Broadcasting Theorem as was presented in 1997 by Buzek et al. These authors presented the No-Broadcasting theorem as a stronger version of No-Cloning Theorem[46]. The laws of quantum mechanics forbid us to clone and broadcast an unknown/arbitrary state with perfect accuracy. In more formal terms, it can be stated that there exists no transformation of the form $\rho_A^{in} \rightarrow \rho_{AB}^{out}$, (in which ρ_{AB} is the combined state of the original and copied state) such that:

$$Tr_B\left(\rho_{AB}^{out}\right) = \rho_A^{in}, \quad Tr_A\left(\rho_{AB}^{out}\right) = \rho_B^{in} \tag{2.1}$$

This result was shown by Barnum et al[20]. The stronger version of this theorem is when $\rho_{AB}^{out} = \rho_A^{in} \otimes \rho_B^{in}$. While Wooters, Zurek had shown that perfectly cloning unknown quantum states that are pure is not possible [15], both the no-cloning and no-broadcasting theorem allows for copying a state that is previously known with perfect accuracy as well as copying two states that are orthogonal to each other with perfect accuracy.

Despite the impossibility of perfect copying in quantum mechanics, it is still possible to conceive of quantum copying machines which satisfactorily meets our practical requirements and which do not completely destroy our input states. Such an operational copying machine can have some relaxed requirements[46]:

- The input and output states given by the density operators are equivalent, which is to say $\rho_A^{out} = \rho_B^{out}$
- It is assumed that all states are copied equally well in the event no prior information is available for the original state. This can be ensured by making the Bures distance (a measure of dissimilarity between two quantum states) between the input and output state operators independent of the input state.

- The inevitable entanglement of the quantum copier with the output states would imply a noise being introduced into the combined original-copied system. Therefore, another assumption that is made here is that the Bures distance between the actual output state of the (original+copied) system and the theoretical ideal output state is independent of the input state.

This process described in the above steps can be interpreted to be a case of imperfect or weak broadcasting, which can be of utility if we wish to copy the quantum information only partially and without destroying our initial quantum states.

2.2 Peres-Horodecki Theorem For $2 \otimes 2$ and $2 \otimes 3$ dimensional systems

The oldest criterion to identify the separability of a bipartite state was given by Peres-Horodecki (**PH**) criterion [54, 55]. The criterion states that if the partial transpose $\rho^T_{m\mu,\eta v} = \rho_{mv,\eta\mu}$ of the joint density matrix ρ of the state is negative, then the state is entangled. Equivalently, it can be said that if at least one of the eigenvalues of a partially transposed density operator for the state ρ is negative, then the state is entangled. This means all separable states have a positive partial transpose. However, in general, the separability condition is only a necessary but not a sufficient criterion. It is necessary and sufficient only in the case of $2 \otimes 2$ and $2 \otimes 3$ dimensional systems. In higher dimensional systems, if partially transposed density operator of the state has a negative eigenvalue, then the state is entangled and is known as the **NPT entangled state**. But in the case when all eigenvalues are positive, there can be no firm conclusion with respect to separability.

Mathematically it is equivalent to the condition that either of the two matrices below have a negative determinant:

$$W_3 = \left(\begin{array}{cc|c} W_2 & & \rho_{00,10} \\ & & \\ & & \rho_{00,11} \\ \hline \rho_{10,00} & \rho_{11,00} & \rho_{10,10} \end{array}\right)$$

$$W_4 = \left(\begin{array}{ccc|c} & W_3 & & \rho_{01,10} \\ & & & \rho_{01,11} \\ & & & \rho_{11,10} \\ \hline \rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11} \end{array}\right)$$

while at the same time W_2's determinant is not negative, where

$$W_2 = \begin{pmatrix} \rho_{00,00} & \rho_{01,00} \\ \rho_{00,01} & \rho_{01,01} \end{pmatrix}$$

It is possible to have entangled states with positive partial transpose. These states are known as **PPT entangled state**. Therefore, in the case of $3 \otimes 3$ dimensional system, which is the subject of this thesis, there can be both PPT as well as NPT entangled states.

In practice, it is often not possible to compute the eigenvalues of a given density operator as it may have multiple argument variables. To overcome this problem, a different separability criterion is used that is based on parameters of Bloch sphere.

Separability criterion with respect to Bloch sphere parameters: The separability criteria mentioned in the previous section can also be represented in terms of the parameters of the Bloch sphere representation of the two systems[47]. This uses the Ky Fan norm of the density matrix of the combined state. For a matrix A belonging to the complex vector space $C^{m\times n}$, its Ky Fan norm is the sum of all the singular values denoted by χ_i

$$\|A\|_{KF} = \sum_{i=1}^{\min\{m,n\}} \chi_i = \text{Tr}\sqrt{A^\dagger A}$$

Once, the Ky Fan norm is calculated, the separability condition for an $M \otimes N$ bipartite state can be stated as:

$$\sqrt{\frac{2(M-1)}{M}}\|X\|_2 + \sqrt{\frac{2(N-1)}{N}}\|Y\|_2 + \sqrt{\frac{4(M-1)(N-1)}{(MN)}}\|T\|_{KF} \leq 1, \tag{2.2}$$

Here $\vec{X}$, $\vec{Y}$ and T are Bloch sphere parameters and $||.||_2$ is the Euclidean norm. If the inequality is satisfied the bipartite state is separable. If however, if the inequality is violated, then nothing can be said about the separability of the state.

2.3 Bound Entangled States

A significant number of mixed entangled states can be purified to yield pure entangled states, which can be used for various quantum information protocols. Nevertheless, some mixed entangled states cannot be purified to extract a pure entangled state, and these types of entangled states are referred to as bound entangled states[37].

It has been observed that entangled states with a positive partial transpose are bound entangled, which is also referred to as undistillable, and the research literature contains several instances of such states[38]. However, it remains unknown whether a state with a negative partial transpose is bound entangled.

Entangled states with positive partial transpose (PPTES), even though being a less powerful form of entanglement, have been employed in quantum information protocols, such as quantum key generation[39]. As a result, generating PPTES is a fascinating problem in quantum information science from both mathematical and physical perspectives. Although there are several mathematically rigorous constructions of PPTES, there are fewer physical constructions available.

The realignment criterion is a straightforward method to detect entanglement in PPT states. It asserts that all separable states, $\rho \in M_m \otimes M_N$ fulfill the condition $||R(\rho)||_{tr} \leq 1$, where R is a linear map called "realignment" map, defined on elementary tensors as $R(|i\rangle\langle j| \otimes |k\rangle\langle l|) = |i\rangle\langle k| \otimes |j\rangle\langle l|$. $||R(\rho)||_{tr} > 1$ therefore implies entangled states.

2.4 Absolute PPT States

A bipartite quantum state $\rho \in \mathcal{H}_n \otimes \mathcal{H}_n$ (a Hilbert space of dimension $n \otimes n$) is said to belong to a set of absolutely PPT states if and only if $\mathcal{U}\rho\mathcal{U}^\dagger$ has positive partial transpose (PPT) for all unitary operators $\mathcal{U} \in \mathcal{H}_n \otimes \mathcal{H}_n$. For the special case of a state $\rho \in \mathcal{H}_3 \otimes \mathcal{H}_n$ to be absolutely PPT, the condition transforms to the eigenvalues ($\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{3n}$) of the state satisfying both the semi-definiteness conditions given below [56]. The absolute PPT states can be both separable and entangled. Distinguishing the two in absolute PPT states is still an open problem.

$$L_1 := \begin{bmatrix} 2\mu_{3n} & \mu_{3n-1} - \mu_1 & \mu_{3n-3} - \mu_2 \\ \mu_{3n-1} - \mu_1 & 2\mu_{3n-2} & \mu_{3n-4} - \mu_3 \\ \mu_{3n-3} - \mu_2 & \mu_{3n-4} - \mu_3 & 2\mu_{3n-5} \end{bmatrix} \geq 0, \tag{2.3}$$

$$L_2 := \begin{bmatrix} 2\mu_{3n} & \mu_{3n-1} - \mu_1 & \mu_{3n-3} - \mu_2 \\ \mu_{3n-1} - \mu_1 & 2\mu_{3n-3} & \mu_{3n-4} - \mu_3 \\ \mu_{3n-2} - \mu_2 & \mu_{3n-4} - \mu_3 & 2\mu_{3n-5} \end{bmatrix} \geq 0. \tag{2.4}$$

2.5 Approximate Quantum Cloning

In previous sections, the no-cloning theorem has already been discussed. The theorem states that for an arbitrary unknown quantum state $|\psi\rangle$, it is impossible to get two copies of the state $|\psi\rangle$ perfectly. In other words, there does not exist any completely positive trace preserving map C such that, $C : |\psi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$, for all $|\psi\rangle$ in the Hilbert space $\mathcal{H}$. However, the theorem never rules out the possibility of approximate cloning. Approximate cloning has been achieved in laboratory experiments and it has a cloning fidelity of $\frac{5}{6}$ which is optimum. In approximate quantum cloning machines (AQCM) [24, 25, 27], the fidelity of cloning is either dependent on the parameters specifying the input state (state dependent quantum cloning machines)[19, 30] or is fixed and independent of the state parameters (state independent or universal cloning machines) [28, 29]. Both of these types of cloning machines are symmetric in the sense that copies at the output port are identical with each other. Beyond symmetric cloners there exist asymmetric cloners having different copies at the output port [32].

This work involves the use of Heisenberg cloning machine, which is described as below:

Heisenberg Cloning Machine

The Heisenberg machine generates a second clone that has the highest possible fidelity for a given fidelity of the first clone. For the cloning of a generalized qudit, the general unitary transformation by this machine is given by:

$$
\begin{aligned}
U|j\rangle_x|00\rangle_{yz} \rightarrow &\sqrt{\frac{2}{d+1}}\Big(|j\rangle_x|j\rangle_y|j\rangle_z \\
+\frac{1}{2}\sum_{r=1}^{d-1}|j\rangle_x|\overline{j+r}\rangle_y|\overline{j+r}\rangle_z &+ \frac{1}{2}\sum_{r=1}^{d-1}|\overline{j+r}\rangle_x|j\rangle_y|\overline{j+r}\rangle_z\Big),
\end{aligned}
\tag{2.5}
$$

The suffixes 'x' and 'y' denote the clones, 'z' denotes the ancillary state, while d is the number of dimensions.

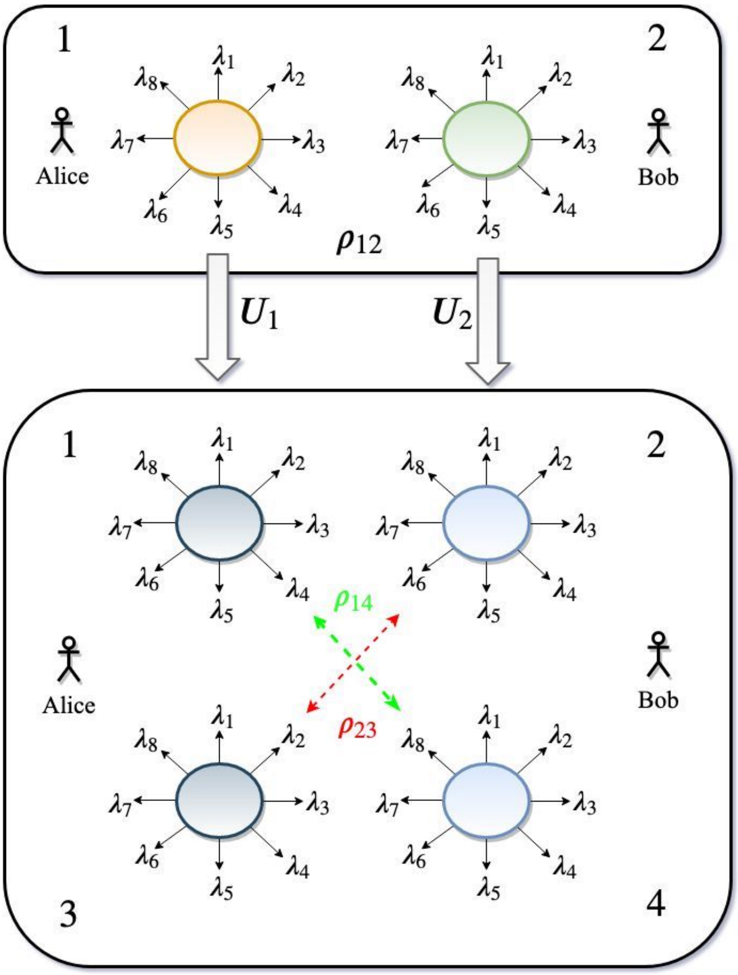

Figure 2.1: A schematic diagram depicting the application of local cloning unitaries U_1 and U_2 on a qutrit-qutrit input state ρ_{12} shared between two hypothetical spacelike separated observers named Alice & Bob to get the non local output states ρ_{14} and ρ_{23}. The qutrit system on both sides is illustrated with a sphere having eight arrows (λ_i) which depits the Gell-Mann matrices.

2.6 Broadcasting of quantum entanglement by cloning

Quantum entanglement is one the key resources that is required for information processing tasks and distributed computing. Thus the aim of distributing entanglement across various nodes in a network is of practical importance. In a network, there is always a requirement of more entangled pairs. This process of creating larger number of entangled pairs with lesser entanglement from a entangled pair with larger entanglement is termed as "broadcasting of quantum entanglement". This chapter is based on using quantum cloning to achieve this task. However there can be several strategies to broadcast entanglement.

Moving beyond $2 \otimes 2$ and $2 \otimes 3$ dimensional systems, the concept of entanglement changes as there can be both **PPT** and **NPT entangled states**. In general, PPT entangled states are not useful in information processing tasks whereas NPT entangled states are useful. In this chapter, broadcasting of NPT entanglement in a qutrit-qutrit system ($3 \otimes 3$) is investigated. Consider two parties Alice and Bob who share a generalized qutrit-qutrit mixed state ρ_{12}(2.8) as initial input state. Local unitary operations for cloning $U_{13} \otimes U_{24}$ on the qutrit pairs $(1,3)$ and $(2,4)$ are then applied. After carrying out partial trace of the subsystems $(2,4)$ and $(1,3)$, one can get the local output states as $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ on Alice's side and Bob's side respectively. Similarly by tracing out the appropriate qutrits, one gets two plausible groups of nonlocal output states $\tilde{\rho}_{14}$ and $\tilde{\rho}_{23}$. The representation of the process is provided in Figure 2.1.

The expression for non-local output states across the subsystems of the two spatially separated parties Alice and Bob are expressed by:

$$\begin{aligned} \tilde{\rho}_{14} &= Tr_{23}[\tilde{\rho}_{1234}] \\ &= Tr_{23}[U_{13} \otimes U_{24}(\rho_{12} \otimes \rho^b_{34} \otimes \rho^m_{56})U^\dagger_{13} \otimes U^\dagger_{24}], \\ \tilde{\rho}_{23} &= Tr_{14}[\tilde{\rho}_{1234}] \\ &= Tr_{14}[U_{13} \otimes U_{24}(\rho_{12} \otimes \rho^b_{34} \otimes \rho^m_{56})U^\dagger_{13} \otimes U^\dagger_{24}], \end{aligned} \tag{2.6}$$

Additionally, the local output states within their individual subsystems are expressed by:

$$\begin{aligned} \tilde{\rho}_{13} &= Tr_{24}[\tilde{\rho}_{1234}] \\ &= Tr_{24}[U_{13} \otimes U_{24}(\rho_{12} \otimes \rho^b_{34} \otimes \rho^m_{56})U^\dagger_{13} \otimes U^\dagger_{24}], \\ \tilde{\rho}_{24} &= Tr_{13}[\tilde{\rho}_{1234}] \\ &= Tr_{13}[U_{13} \otimes U_{24}(\rho_{12} \otimes \rho^b_{34} \otimes \rho^m_{56})U^\dagger_{13} \otimes U^\dagger_{24}]. \end{aligned} \tag{2.7}$$

Here the cloning operations U_1 and U_2 are optimal universal asymmetric Heisenberg cloning transformations. The states $\rho^b_{34} = |00\rangle\langle 00|$ and $\rho^m_{56} = |00\rangle\langle 00|$ represent the initial blank state and the initial machine state respectively. In order to attain the objective to broadcast **NPT** entanglement between desired pairs $(1,4)$ and $(2,3)$, it should be possible to create **NPT** entanglement between nonlocal pairs $(1,4)$ and $(2,3)$ irrespective of the local pairs $(1,3)$ and $(2,4)$.

A summary of the previous contributions in this direction, in contrast to those in this chapter is explicitly presented in Table (2.1).

Table 2.1: Summary of earlier results along with those in the present work on broadcasting of entanglement, discord and coherence. The abbreviations such as NME, MEMS, TPCS, IS, 2-qubit general, qubit-qutrit general and qubit-qudit general stand for non-maximally entangled state, maximally entangled mixed state, two parameter class of states, isotropic states, general two qubit mixed state, general qubit-qutrit mixed state, and general qubit-qudit mixed state classes respectively.

System's dimension	**Resource state**	**Broadcasting of**	**Cloning operation**	**Author(s)**
$2 \otimes 2$	NME	Entanglement	Symmetric	Buzek *et al.* and Hillery [46, 25]
$2 \otimes 2$	NME	Entanglement	Symmetric	Bandyopadhyay *et al.* [50]
$2 \otimes 2$	NME	Entanglement	Asymmetric	Ghiu [32]
$2 \otimes 2$	2-qubit general	Entanglement and Discord	Symmetric	Chatterjee *et al.* [49]
$2 \otimes 2$	2-qubit general	Entanglement and Discord	Asymmetric	Jain *et al.* [51]
$2 \otimes 2$	2-qubit general	Coherence	Symmetric	Sharma *et al.* [52]
$2 \otimes 3$	qubit-qutrit general	Entanglement	Symmetric	[53]
$2 \otimes$ d	qubit-qudit general	Discord, Coherence	Symmetric	[53]
$3 \otimes 3$	TPCS, IS	Entanglement	Symmetric	This work

2.7 Broadcasting Of Entanglement In $3 \otimes 3$ Dimension

A general qutrit-qutrit mixed entangled state ρ_{12} (which is the resource state) can be canonically represented as:

$$\rho_{12} = \frac{1}{9}\Bigg(\mathbb{I}_3 \otimes \mathbb{I}_3 + \sum_{i=1}^{8} x_i G_i \otimes \mathbb{I}_3 + \sum_{j=1}^{8} y_j \mathbb{I}_3 \otimes G_j + \sum_{i=1}^{8}\sum_{j=1}^{8} t_{ij} G_i \otimes G_j\Bigg) = \{\vec{X}, \vec{Y}, T\}, \tag{2.8}$$

where $x_i = Tr[\rho_{12}(G_i \otimes \mathbb{I}_3)]$, $y_j = Tr[\rho_{12}(\mathbb{I}_3 \otimes G_j)]$, $T_{ij} = Tr[\rho_{12}(G_i \otimes G_j)]$, G_j's are Gell-Mann matrices and $\vec{X}$, $\vec{Y}$ and T are the Bloch vectors and the correlation matrix respectively. $\mathbb{I}_3$ represents identity matrix of order 3×3.

Here the focus would be on two mixed entangled states in $3 \otimes 3$ dimensions, namely : (A) Two Parameter Class of States (TPCS) and (B) Isotropic state. The question is whether it is possible to broadcast the NPT entanglement present in these two states.

2.7.1 Two Parameter Class of States (TPCS)

Consider the following class of states with two real parameters b and c in bipartite qutrit quantum systems.

$$\rho_{b,c} = a\sum_{i=0}^{2}|ii\rangle\langle ii| + b\sum_{i,j=0,i<j}^{2}|\psi_{ij}^{-}\rangle\langle\psi_{ij}^{-}| + c\sum_{i,j=0,i<j}^{2}|\psi_{ij}^{+}\rangle\langle\psi_{ij}^{+}|, \tag{2.9}$$

where $|\psi_{ij}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$ and $\{|i\rangle\}$ represents the vectors in the computational basis.[57] The parameter a is dependent on parameters b and c by unit trace condition, $3\ (a + b + c) = 1$. From unit trace condition, b and c can vary from 0 to $\frac{1}{3}$. However for the matrix $\rho_{b,c}$ to remain positive semi-definite, the parameter 'c' can take values at most $\frac{1}{3} - b$. This state is NPT entangled when state parameter 'b' ranges from $\frac{1}{6}$ to $\frac{1}{3}$. It's also worth noting that the states described by points on the line $3b + 6c = 1$ correspond to qutrit Werner states. A Werner state is a quantum state that is bipartite with $d \times d$ dimensions and remains unchanged under $U \otimes U$ for any unitary operator U

This input state ρ_{12} is shared by two parties, Alice and Bob. Both parties apply local cloning machine given in eqn (4) to obtain a composite system $\tilde{\rho}_{1234}$. By tracing out the appropriate qutrits on both sides, the two plausible groups of non local states are obtained as follows :

$$\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{\vec{X}_{tpcs}, \vec{Y}_{tpcs}, T_{tpcs}\right\}, \tag{2.10}$$

where $\vec{X}_{tpcs} = \vec{Y}_{tpcs} = 0,0,0,0,0,0,0,0$ and the non zeros entries of the correlation matrix (T_{tpcs}) are $t_{1,1} = \frac{25(c-b)}{64}$, $t_{2,2} = \frac{25(c-b)}{64}$, $t_{3,3} = \frac{25(2-9b-9c)}{192}$, $t_{4,4} = \frac{25(c-b)}{64}$, $t_{5,5} = \frac{25(c-b)}{64}$, $t_{6,6} = \frac{25(c-b)}{64}$, $t_{7,7} = \frac{25(c-b)}{64}$, $t_{8,8} = \frac{25(2-9b-9c)}{192}$. Here $t_{i,j}$ denotes the element of i^{th} row and j^{th} column of the correlation matrix. Peres criterion can be applied to determine the condition when these nonlocal output state are NPT entangled or not. It is observed that the output states are NPT entangled when state parameter b is greater than $\frac{19}{75}$ and less than $\frac{1}{3}$ which is a subset of the initial parameter range $[1/6, 1/3]$. An explicit example of an NPT entangled TPCS state that can be broadcast and still remain NPT entangled is $\frac{4}{15\sqrt{2}}[(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) + (|02\rangle - |20\rangle)(\langle 02| - \langle 20|) + (|12\rangle - |21\rangle)(\langle 12| - \langle 21|)] + \frac{1}{15\sqrt{2}}[(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) + (|02\rangle + |20\rangle)(\langle 02| + \langle 20|) + (|12\rangle + |21\rangle)(\langle 12| + \langle 21|)]$ with the value of $b = \frac{4}{15}$ and $c = \frac{1}{15}$.

To demonstrate the generation of absolute PPT states in non local output states (ρ_{14} and ρ_{23}), 10^4 random values of state parameters b and c from a uniform random distribution are generated and it is observed that absolute PPT were found for some state parameters. This is displayed in figure (2.2). The states are represented by brown dots.

An example of an NPT entangled TPCS state that can be broadcast to produce ABPPT states is $\frac{2}{15}(|00\rangle\langle 00|+|11\rangle\langle 11|+|22\rangle\langle 22|)+\frac{1}{5}[(|01\rangle-|10\rangle)(\langle 01|-\langle 10|)+(|02\rangle-|20\rangle)(\langle 02|-\langle 20|)+(|12\rangle-|21\rangle)(\langle 12|-\langle 21|)]$ with the value of $b=\frac{1}{5}$ and $c=0$.

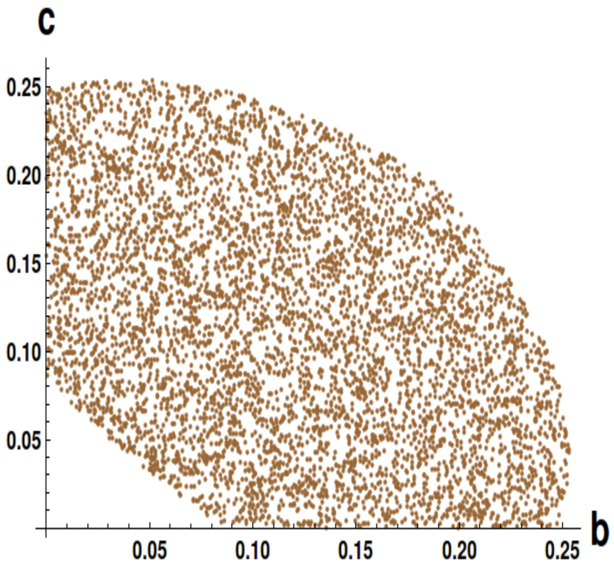

Figure 2.2: Plot depicting the values (in brown) of two input state parameters: b and c of the state ρ_{12}, which will generate the absolute PPT states in ρ_{14} and ρ_{23}.

2.7.2 Isotropic States (IS)

In this subsection, a class of density matrices, called as isotropic density matrices is considered, which are convex mixture of a maximally entangled state and the maximally mixed state :

$$\rho_f = \frac{1-f}{d^2-1}(\mathbb{I} - |\psi^+\rangle\langle\psi^+|) + f|\psi^+\rangle\langle\psi^+|, \tag{2.11}$$

for $0 \leq f \leq 1$ and $|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. These states are separable for $f \leq \frac{1}{d}$, and entangled otherwise. For $d=3$ case, these states are NPT entangled when state parameter 'f' is greater than $\frac{1}{3}$. Let us assume that this isotropic state is shared by two parties, Alice and Bob. They both apply local cloning transformations given by eqn (4) on their respective qutrits to obtain the composite system $\tilde{\rho}_{1234}$. By tracing out the ancillas and local qutrits on both sides, the nonlocal output states can be obtained as follows,

$$\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{\vec{X}, \vec{Y}, T\right\}, \tag{2.12}$$

where $\vec{X} = \vec{Y} = \{0,0,0,0,0,0,0,0\}$ and the non zero entries of correlation matrix (T) are $t_{1,1} = \frac{25(-1+9f)}{768}$, $t_{2,2} = -\frac{25(-1+9f)}{768}$, $t_{3,3} = \frac{25(-1+9f)}{768}$, $t_{4,4} = \frac{25(-1+9f)}{768}$, $t_{5,5} = -\frac{25(-1+9f)}{768}$, $t_{6,6} = \frac{25(-1+9f)}{768}$, $t_{7,7} = -\frac{25(-1+9f)}{768}$, $t_{8,8} = \frac{25(-1+9f)}{768}$. Here $t_{i,j}$ denotes the element of i^{th} row and j^{th} column of the correlation matrix. Now, Peres criterion can again be applied to determine the condition when these nonlocal output state are NPT entangled for non-optimal broadcasting. It is observed that the output states are NPT entangled when state parameter f is greater than $\frac{17}{25}$ and less than 1. An example of an NPT entangled isotropic state which will remain NPT entangled on broadcasting would be $\frac{1}{40}\mathbb{I} + \frac{31}{40}|\psi^+\rangle\langle\psi^+|$ with the value of f being $\frac{4}{5}$.

It can also be observed that absolute PPT were found in nonlocal output states (ρ_{14} and ρ_{23}) when the range of state parameter 'f' is less than $\frac{433}{825}$. An example of an NPT entangled isotropic state that can be broadcast to produce ABPPT states is $\frac{1}{16}\mathbb{I} + \frac{59}{80}|\psi^+\rangle\langle\psi^+|$ with the value of f being $\frac{1}{2}$.

2.8 Conclusion

In conclusion what can be said about the work presented in this thesis is that it is possible to broadcast NPT entangled states in $3 \otimes 3$ system (TPCS,IS) for certain range of input state parameters. This can be considered a significant step to broadcast NPT entanglement beyond $2\otimes 2$ and $2\otimes 3$ systems. Another significant aspect of this submission is that a physical procedure is given as a derivative of which ABPPT states can be derived as an output. This work initiates the process of investigating broadcasting of NPT entangled states in higher dimension and broadcasting of other quantum resources like correlation and coherence in $3 \otimes 3$ system.

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