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CORRELATION BETWEEN DIAGONAL RATIO AND CONDITION NUMBER OF THE GENERALIZED INERTIA MATRIX OF A SERIAL-CHAIN

The condition number of the Generalized Inertia Matrix (GIM) of a serial chain can be used to measure its ill-conditioning. However, computation of the condition number is computationally very expensive. Therefore, this paper investigates alternative means to estimate the condition number, in particular, for a very long serial-chain. For this, the diagonal elements of the GIM are examined. It is found that the ratio of the largest and smallest diagonal elements of the GIM, when scaled using an initial estimate of the condition number, closely resembles the condition number. This significantly simplifies the process of detecting ill-conditioning of the GIM, which may help to decide on stability of the system at hand.

1. Introduction

The GIM of a multibody system is a function of its joint variables and plays a vital role in simulation and control. It is interesting to note that ill-conditioning of the GIM results into loss of accuracy, mainly, in forward dynamics [1] and poor control performance of the joints [2]. Therefore, the condition number is used as an important measure to quantify ill-conditioning of the GIM [2]. If norm-2 definition [3] is used, the condition number is defined as the ratio of the largest and smallest singular values. As the GIM is a symmetric and positive-definite matrix, its condition number is nothing else but the ratio of the maximum and minimum eigenvalues [3]. Even though the condition number is widely used as a mean to judge ill-conditioning of the GIM, its computation using eigenvalues is computationally very expensive.

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As a result, if the health of the GIM can be judged using any alternative property of the GIM, significant computational savings can be obtained. Such alternative means were seldom reported in literature to the best of our knowledge.

Interestingly, in 1975 Mitra and Klein [4] showed that the pivot ratio, defined as ratio of highest to smallest pivot element, can be used to predict instability. They used the concept in integral equations of electromagnetics. However, it was found during this work, that the use of pivot cannot provide a good estimate of the trend of the condition number. The facts 1) the trace of GIM is equal to the sum of eigenvalues, and 2) each diagonal element of the GIM carries the knowledge of the system ahead of the body corresponding to the index of the diagonal element, motivated us to use the ratio of the highest to smallest diagonal elements of the GIM as a measure of ill-conditioning of the GIM. This ratio will be referred to as diagonal ratio hereafter for the sake of simplicity. As the elements of the GIM are readily available as a by-product of either inverse or forward dynamics algorithms [1], no additional computation is required to calculate the diagonal ratio. The diagonal ratio was compared to the condition number as a mean to judge ill-conditioning. Later, the notion of scaled diagonal ratio is introduced.

Rest of the paper is organized as follows: Ill-conditioning of the GIM is introduced in Section 2. Some important properties of the GIM are presented in Section 3 and several numerical illustrations are provided in Section 4. Finally, conclusions are given in Section 5.

2. Ill-conditioning of the GIM

According to [5], the equations of motion of a tree-structured multibody system may be represented as

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \boldsymbol{\tau} \tag{1}$$

where $\mathbf{I}$, $\mathbf{q}$, $\mathbf{C}$, and $\boldsymbol{\tau}$ represent the GIM, vector of generalized coordinates, matrix of convective inertia terms, and vector of the generalized external forces, respectively.

As shown in [2], the ill-conditioning of the GIM not only affects the accuracy of simulation results but also the control performance of a system. Hence the measure of ill-conditioning may help to take any corrective measures during simulation or control. This, however, is beyond the scope of the paper; rather we focus here on efficient estimation of ill-conditioning, which may be used as a guide. Here, we use simulation only to demonstrate how the condition of the GIM varies over time.

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Simulation of a multibody system consists of 1) solution of a system of algebraic equations linear in joint accelerations, and 2) numerical integration. The joint accelerations, denoted with $\ddot{\mathbf{q}}$, are obtained from Eq. (1) as

$$\ddot{\mathbf{q}} = \mathbf{I}^{-1}\boldsymbol{\varphi}, \text{ where } \boldsymbol{\varphi} = \boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} \tag{2}$$

Note that the explicit inversion of the GIM, **I**, is not required to solve for $\ddot{\mathbf{q}}$ as the GIM can be factorized using **LU** or Cholesky decomposition [3], and then $\ddot{\mathbf{q}}$ is calculated by backward and forward substitutions [3]. However, when the GIM becomes ill-conditioned, small perturbations in the system can produce relatively large changes in the solutions. Ill-conditioning of a matrix is defined as the closeness of a matrix to its singularity [3]. For small change in right hand side of Eq. (2), the solution is disturbed according to

$$(\ddot{\mathbf{q}} + \boldsymbol{\delta}\ddot{\mathbf{q}}) = \mathbf{I}^{-1}(\boldsymbol{\varphi} + \delta\boldsymbol{\varphi}), \tag{3}$$

Resulting in the relative error [3]

$$\frac{\|\boldsymbol{\delta}\ddot{\mathbf{q}}\|}{\|\ddot{\mathbf{q}}\|} \leq \left\|\mathbf{I}^{-1}\right\| \|\mathbf{I}\| \frac{\|\boldsymbol{\delta\varphi}\|}{\|\boldsymbol{\varphi}\|} \tag{4}$$

where $\|\mathbf{I}\|$ represents the norm of the GIM, and $\kappa(\mathbf{I}) = \left\|\mathbf{I}^{-1}\right\| \|\mathbf{I}\|$ is defined as its condition number which determines the amount by which the solution $\ddot{\mathbf{q}}$ gets magnified for a small change in the right hand side, i.e., $\boldsymbol{\phi}$. If the condition number of the GIM is very high, it is ill-conditioned or close to singularity. If we choose norm-2 [3], then the condition number of the GIM is found from

$$\kappa_2(\mathbf{I}) = \frac{\sigma_{max}(\mathbf{I})}{\sigma_{min}(\mathbf{I})} \tag{5}$$

where $\sigma_{max}(\mathbf{I})$ and $\sigma_{min}(\mathbf{I})$ are the maximum and minimum singular values of the GIM. As the GIM is symmetric and positive definite, its singular values are nothing else but the eigenvalues, and Eq. (5) can be rewritten as

$$\kappa_2(\mathbf{I}) = \frac{\lambda_{max}}{\lambda_{min}} \tag{6}$$

where λ_{max} and λ_{min} are the highest and smallest eigenvalues of the GIM.

It is worth noting that serial multibody systems with identical links or homogenous rods have worst condition numbers in the order of $O(4n^4)$ [1] where n is the total number of links. Hence, with the increase in n there is a high chance of loss of accuracy in the computation of the joint accelerations. As a result, a numerical integrator may require small step sizes in order to provide accurate solution. This phenomenon is also known as numerical

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stiffness [6]. In order to get some idea of ill-conditioning, a 4-link serial chain with identical links moving under gravity is considered as shown in Fig. 1. Each link is assumed to be a slender homogeneous rod with mass $m = 2.2$ kg and length $l = 1$m. The GIM $\mathbf{I}$ and the forces $\boldsymbol{\varphi}$ for the configuration $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ are given by

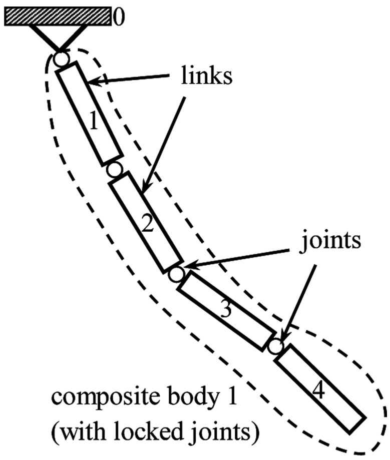

Fig. 1. A 4-link chain with only revolute joints

$$\mathbf{I} = \begin{bmatrix} 47.0822 & & \text{sym} & \\ 29.7942 & 19.8629 & & \\ 14.7132 & 10.2993 & 5.8854 & \\ 4.0462 & 2.9427 & 1.8392 & 0.7357 \end{bmatrix}, \text{ and } \boldsymbol{\varphi} = \begin{bmatrix} -86.6015 \\ -48.7133 \\ -21.6504 \\ -5.4126 \end{bmatrix} \tag{7}$$

where '*sym*' denotes symmetric elements of the GIM. The solution of the joint accelerations can be obtained as

$$\ddot{\mathbf{q}} = \mathbf{I}^{-1}\varphi = \begin{bmatrix} -6.2189 \\ 7.8864 \\ -2.1226 \\ 0.6071 \end{bmatrix} \tag{8}$$

Note that the condition number of the GIM in Eq. (7) is $\kappa_2(\mathbf{I}) = 1074$, which is rather high for such a small system. In order to see the effect of small

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perturbations of $\boldsymbol{\varphi}$ on $\ddot{\mathbf{q}}$, small deviations due to rounding error in $\boldsymbol{\varphi}$ are considered as

$$\boldsymbol{\varphi} = \begin{bmatrix} -86.0 \\ -48.7 \\ -21.6 \\ -5.4 \end{bmatrix}, \text{ resulting in } \ddot{\mathbf{q}} = \mathbf{I}^{-1}\boldsymbol{\varphi} = \begin{bmatrix} -5.7477 \\ 6.7586 \\ -1.2128 \\ 0.2696 \end{bmatrix}. \tag{9}$$

Thus, the small changes in $\varphi_1, \varphi_2, \varphi_3$ and φ_4 of 0.69%, 0.02%, 0.23% and 0.23%, respectively, result in relative high percentage changes in accelerations $\ddot{q}_1$, $\ddot{q}_2$, $\ddot{q}_3$, and $\ddot{q}_4$ of 7%, 14%, 42% and 55%, respectively, which are significant. This change will be even more significant in large systems. Hence, an estimate of ill-conditioning of the GIM is very essential.

It was shown in [5, 7] that the GIM, obtained using the concept of the Decoupled Natural Orthogonal Complement (DeNOC) matrices, stores the information of mass and inertia properties in a very systematic manner. Hence, study of the elements of the GIM may provide thorough insight about ill-conditioning. Therefore, some important characteristic of the GIM are discussed next.

3. Characteristics of the GIM

The GIM of a serial chain has the following representation:

$$\mathbf{I} \equiv \begin{bmatrix} I_{11} & & & & sym \\ I_{21} & I_{22} & & & \\ I_{31} & I_{32} & I_{33} & & \\ \vdots & \vdots & \ddots & \ddots & \\ I_{n1} & I_{n2} & \cdots & I_{nn-1} & I_{nn} \end{bmatrix}. \tag{10}$$

where I_{ij} represents the (i,j)-th element of the GIM. The analytical expression of the (i,j)-th element of the GIM is given by [7]

$$\begin{aligned} I_{ii} &= \mathbf{p}_i^T \tilde{\mathbf{M}}_i \mathbf{p}_i \\ I_{ij} &= \mathbf{p}_i^T \tilde{\mathbf{M}}_i \mathbf{A}_{i,j} \mathbf{p}_j \equiv I_{ji} \end{aligned} \tag{11}$$

In Eq. (11), $\mathbf{A}_{i,j}$ and $\mathbf{p}_j$ are the twist-propagation matrix and motion propagation vectors [7], respectively, whereas $\tilde{\mathbf{M}}_i$ is the mass matrix of composite body which contains the mass and inertia properties of the system comprising of all rigidly connected links upstream of the i^{th} link including itself. It is obtained from mass matrix $\mathbf{M}_i$ of link i as

$$\tilde{\mathbf{M}}_i = \mathbf{M}_i + \mathbf{A}_{j,i}^T \tilde{\mathbf{M}}_j \mathbf{A}_{j,i} \tag{12}$$

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where for the terminal link $\tilde{\mathbf{M}}_n = \mathbf{M}_n$. The structure of the mass matrix of a composite body may vary with the choice of independent generalized coordinates. However, the present choice is based on a popular choice for the serial-type systems, i.e., relative coordinates.

It is worth noting that the GIM is a positive definite matrix, and hence, the diagonal terms are always greater than zero, i.e., $\mathbf{p}_i^T\tilde{\mathbf{M}}_i\mathbf{p}_i > 0$ for $i = 1, \ldots, n$. Using the analytical expressions in Eq. (11), the GIM of the 4-link planar chain, shown in Fig. 1, is obtained as

$$\mathbf{I} = \begin{bmatrix} \mathbf{p}_1^T\tilde{\mathbf{M}}_1\mathbf{p}_1 & & & sym \\ \mathbf{p}_2^T\tilde{\mathbf{M}}_2\mathbf{A}_{21}\mathbf{p}_1 & \mathbf{p}_2^T\tilde{\mathbf{M}}_2\mathbf{p}_2 & & \\ \mathbf{p}_3^T\tilde{\mathbf{M}}_3\mathbf{A}_{31}\mathbf{p}_1 & \mathbf{p}_3^T\tilde{\mathbf{M}}_3\mathbf{A}_{32}\mathbf{p}_2 & \mathbf{p}_3^T\tilde{\mathbf{M}}_3\mathbf{p}_3 & \\ \mathbf{p}_4^T\tilde{\mathbf{M}}_4\mathbf{A}_{41}\mathbf{p}_1 & \mathbf{p}_4^T\tilde{\mathbf{M}}_4\mathbf{A}_{42}\mathbf{p}_2 & \mathbf{p}_4^T\tilde{\mathbf{M}}_4\mathbf{A}_{43}\mathbf{p}_3 & \mathbf{p}_4^T\tilde{\mathbf{M}}_4\mathbf{p}_4 \end{bmatrix} \tag{13}$$

In Eq. (14), $\tilde{\mathbf{M}}_4 = \mathbf{M}_4$ represents the mass and inertia properties of the 4^{th} link only, whereas, $\tilde{\mathbf{M}}_1$ represents the mass and inertia properties of all the links, enclosed by the dotted line in Fig. 1. Therefore, the term $\mathbf{p}_1^T\tilde{\mathbf{M}}_1\mathbf{p}_1$ is larger than any other diagonal term and $\mathbf{p}_4^T\tilde{\mathbf{M}}_4\mathbf{p}_4$ is the smallest of all. This is also evident from Eq. (7). Moreover, it obvious that with the increase in the number of links, the term $\mathbf{p}_1^T\tilde{\mathbf{M}}_1\mathbf{p}_1$ will become larger and larger, whereas $\mathbf{p}_n^T\tilde{\mathbf{M}}_n\mathbf{p}_n$ will remain unaffected. Moreover, the 'trace' of the GIM, i.e., the sum of the diagonal elements, is related to the eigenvalues [3] by

$$tr(\mathbf{I}) = \sum_{i=1}^{n} I_{ii} = \sum_{i=1}^{n} \lambda_i, \text{ where } I_{ii} = \mathbf{p}_i^T\tilde{\mathbf{M}}_i\mathbf{p}_i \tag{14}$$

The above two facts motivated us to compare the ratio of the smallest and highest eigenvalues, i.e., condition number in Eq. (6), with the ratio of the largest to smallest diagonal elements of the GIM. These ratios are compared using several numerical examples in the next section.

4. Numerical Illustrations

As introduced in Section 1, the diagonal ratio is defined as the ratio of the largest to smallest diagonal elements of the GIM and will be denoted as $\delta(\mathbf{I}) = I_{11}/I_{nn}$ hereafter. The eigenvalues and the diagonal elements of the GIM for the swinging 4-link chain are plotted in Figs. 2(a-b). It can be seen that the element $I_{11} > (I_{22}, I_{33}, I_{44})$ follows the trend of the highest eigenvalue λ_1 throughout the simulation period. Since, I_{44} is the smallest element, $\delta(\mathbf{I}) = I_{11}/I_{44}$ forms the diagonal ratio, Fig. 2(d). Comparison with the condition

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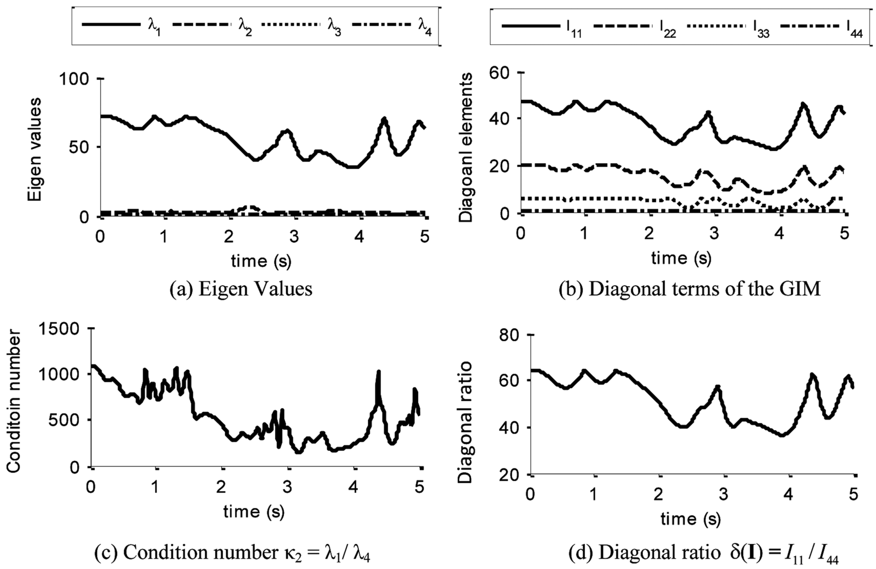

(a) Eigen Values

(b) Diagonal terms of the GIM

(c) Condition number $\kappa_2 = \lambda_1/\lambda_4$

(d) Diagonal ratio $\delta(\mathbf{I}) = I_{11}/I_{44}$

Fig. 2. Different properties of the GIM of a 4-link chain

number $\kappa_2 = \lambda_1/\lambda_4$ in Fig. 2(c) makes clear that the diagonal ratio captures the trend of the condition number.

$$\delta(\mathbf{I}) = I_{11}/I_{44}$$

Next, the diagonal ratio and the condition number of 10- and 20-link serial-chains of identical links are compared as shown in Fig. 3. It is evident from Fig. 3 that with the increase in the number of links the maximum condition number increases and so is the diagonal ratio. Moreover, the diagonal ratio is able to capture the trend of the condition number for both 10- and 20-link chains. It is also evident from Figs. 2(c), 3(a) and 3(c) that the worst condition number for a serial-chain with identical links is about of $O(4n^4)$ [1].

It was observed in [4], that the pivot ratio, the ratio of largest to smallest pivot element obtained from Gaussian elimination [3] of the GIM, can be used as a measure of ill-conditioning. Motivated by this fact pivot ratios of both 10- and 20-link chains are also shown in Fig. 4, which clearly demonstrate that the pivot ratio does not capture the trend of the condition number. Hence, pivot ratio should not be used as a measure of the ill-conditioning of the GIM for serial-chain systems.

Even though the condition number follows the diagonal ratio, their magnitudes are not comparable. In order to have the estimate of the magnitude of the condition number, the notion of scaled diagonal ratio is introduced. The scaled diagonal ratio is nothing else but the diagonal ratio scaled by the

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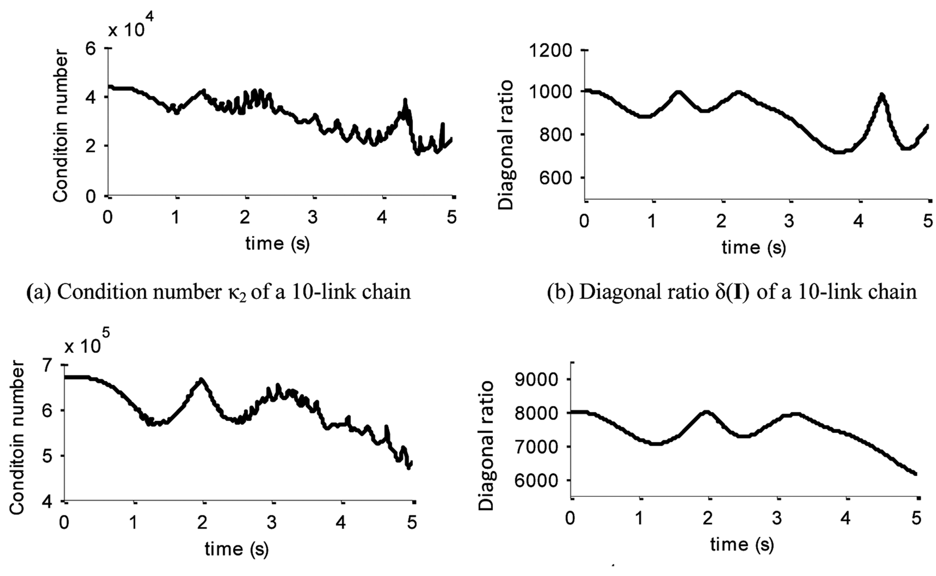
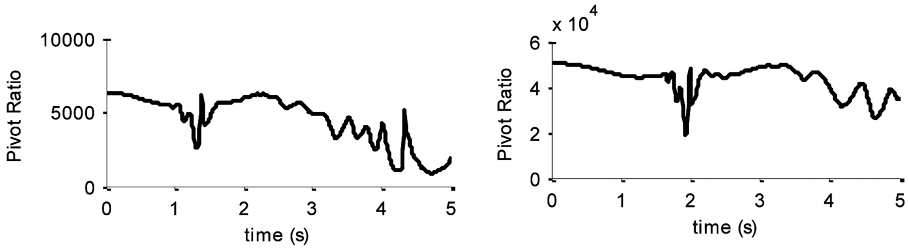


(a) Condition number κ_2 of a 10-link chain (b) Diagonal ratio $\delta(\mathbf{I})$ of a 10-link chain

(c) Condition number κ_2 of a 20-link chain (d) Diagonal ratio $\delta(\mathbf{I})$ of a 20-link chain

Fig. 3. Comparison of the condition number and diagonal ratios for two serial chains



(a) 10-link chain (b) 20-link chain

Fig. 4. Pivot ratios for 10- and 20- link chains

condition number of a system at time t=0:

$$\delta_s(\mathbf{I}) = \alpha\delta(\mathbf{I}) \text{ where } \alpha = \frac{\kappa(\mathbf{I})|_{t=0}}{\delta(\mathbf{I})|_{t=0}} \tag{15}$$

The results in Fig. 5 show that $\delta_s(\mathbf{I})$ not only captures the trend of the condition number but also provides a good estimate of the condition number. This shows that the scaled diagonal ratio $\delta_s(\mathbf{I})$ can be used to estimate ill-conditioning of the GIM during simulation without incurring expensive computation of the condition number.

5. Conclusions

This paper presents a novel method to estimate the ill-conditioning of the GIM during simulation. This method uses the ratio of the largest to smallest

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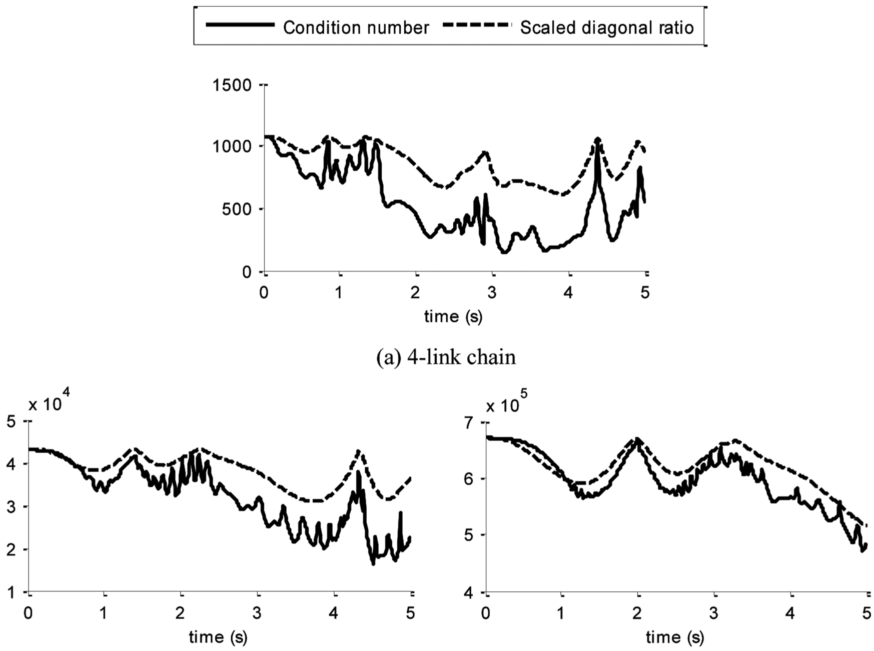

(a) 4-link chain

(b) 10-link chain (c) 20-link chain

Fig. 5. Condition number and scaled diagonal ratio for the 4-, 10- and 20-link chains

diagonal elements of the Generalized Inertia Matrix (GIM) scaled by constant factor. The effectiveness of this method is shown using several numerical examples. The diagonal ratio captures the trend of the condition number, and when scaled with the help of the initial values of the condition number, the resulting scaled diagonal ratio provides magnitude of the condition number, thereby making a very safe decision about the ill-conditioning of the GIM.

The proposed methodology not only makes the estimation of ill-conditioning simple and efficient, but also lends its utility in taking corrective control measures in order to improve control performance and setting adaptive tolerances for the forward dynamics problem, which will be carried out as a future work.

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REFERENCES

[1] Featherstone, R.: An Empirical Study of the Joint Space Inertia Matrix. International Journal of Robotics Research, Vol. 23, No. 9, pp. 859-871, 2004.
[2] Shen Y., Featherstone R.: The Effect of Ill-conditioned Inertia Matrix on Controlling Robot Manipulator. Australasian Conference on Robotics and Automation (ACRA), 2003.
[3] Watkins D.S.: Fundamentals of Matrix Computations, Wiley Interscience, New York, 2002.
[4] Mittra R., Klein C. A.: Use of Pivot Ratios as a Guide to Stability of Matrix Equations Arising in the Method of Moments. IEEE Transactions on Antennas and Propagation, Vol.23, No. 3, pp. 448-450, 1975.
[5] Saha, S. K.: "Dynamics of Serial Multibody Systems Using the Decoupled Natural Orthogonal Complement Matrices," ASME J. of Applied Mechanics, 66, 986-996, 1999.
[6] Shampine L.F., Gear C.W.: A User's View of Solving Stiff Ordinary Differential Equations. SIAM Review, Vol. 21, No. 1, 1979.
[7] Saha S.K.: Analytical Expression for the Inverted Inertia Matrix of Serial Robots. Int. J. of Robotic Research, Vol. 18, No. 1, pp. 116-124, 1999.

Korelacja między współczynnikiem diagonalnym a współczynnikiem uwarunkowania uogólnionej macierzy bezwładności łańcucha szeregowego

S t r e s z c z e n i e

Wskaźnik uwarunkowania jest wykorzystywany jako miara złego uwarunkowania macierzy, np. dla uogólnionej macierzy bezwładności (GIM) łańcucha szeregowego. Niemniej, wyznaczenie tego współczynnika wymaga znacznego nakładu mocy obliczeniowej. Tak więc, w artykule zaproponowano sposoby alternatywne, pozwalające estymować współczynnik uwarunkowania, w szczególności dla bardzo długiego łańcucha szeregowego. W tym celu bada się elementy diagonalne uogólnionej macierzy bezwładności. Wykazano, że stosunek diagonalny (stosunek największego do najmniejszego elementu na głównej przekątnej macierzy bezwładności), przeskalowany przy użyciu estymatora początkowej wartości wskaźnika uwarunkowania, ma wartość bardzo zbliżoną do rzeczywistego wskaźnika uwarunkowania. Jego zastosowanie upraszcza w znaczący sposób ocenę złego uwarunkowania macierzy, dzięki czemu można od razu zdecydować czy układ jest stabilny.

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