Classification of three qubit pure states

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Chandan Dutta, Satyabrata Adhikari, Arpan Das, Pankaj Agrawal, European Physical Journal D 72, 157 (2018)

Classification under SLOCC

Multipartite states are classified according to their inter-convertibility under stochastic local operations and classical communication (SLOCC).

W. Dur, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).

(i) Same quantum information task: Two different entangled states may be used to implement the same task identically if they are equivalent under local unitary (LU).

(ii) Different quantum information task: Two different entangled states used to implement the different task if they are equivalent under SLOCC.

S. M. Zangi, Jun-Li Li, Cong-Feng Qiao, Phys. Rev. A 97, 012301 (2018).

The classification of states with specific symmetries were studied by effective methods under SLOCC.

T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. 103, 070503 (2009).

Few proposals on classification

Proposal for the introduction of special linear group invariant polynomials on n qudits that can be used to classify almost all SLOCC classes.

G. Gour and N. R. Wallach, Phys. Rev. Lett. 111, 060502 (2013)

Proposol for the classification of multipartite states has been given in terms of the maximum degree of non-locality they can exhibit under any choice of local observables.

S. Abramsky, C. Constantin, EPTCS 171, 10 (2014).

Classification of three-qubit under SLOCC

For three qubits, there are only 6 inequivalent SLOCC classes. While for four or more qubits, there are already uncountable number of inequivalent SLOCC classes.

W. Dur, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
H.J. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001);
R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).

This is the indication of the complication that will appear in classifying the SLOCC classes, as the number of qubit increases.

Classification of three-qubit states were studied based on their three-qubit and reduced two-qubit entanglements.

C. Sabín, G. G-Alcaine, Eur. Phys. J. D 48, 435-442 (2008).

Six classes of pure three-qubit states

Separable State: One Class

$(A - B - C)$: $|000>$

Biseparable State: Three Classes

A-BC: |010> + |001>

B-CA: |100> + |001>

C-AB: |010> + |100>

Genuine Entangled State: Two Classes

W state: |100> + |010> + |001>

GHZ state: |000> + |111>

Tangle for pure state

Tangle for three-qubit pure state is defined as

$$\tau = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2$$

$C_{A(BC)}$ refers to the entanglement of qubit A with the joint state of qubits B and C.

C_{AB} denote the concurrence of the entangled state between the qubits A and B.

C_{AC} denote the concurrence of the entangled state between the qubits A and C.

Ref.: V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).

Any three-qubit pure state can be written in the canonical form as

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

$$\lambda_i \geq 0, \quad \sum \lambda_i^2 = 1, \quad \theta \in [0, \pi]$$

Ref.: A. Acin, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001).

Tangle for the state $|\psi\rangle_{ABC}$: $\tau_\psi = 4\lambda_0^2\lambda_4^2$

Mermin operator and newly defined operators

Let us choose the operators $O,\ O_1,\ O_2,\ O_3$ as

$$O = 2\,(\sigma_x \otimes \sigma_x \otimes \sigma_x)$$
$$O_1 = 2\,(\sigma_x \otimes \sigma_x \otimes \sigma_z)$$
$$O_2 = 2\,(\sigma_x \otimes \sigma_z \otimes \sigma_x)$$
$$O_3 = 2\,(\sigma_z \otimes \sigma_x \otimes \sigma_x)$$

How to choose these operators?

The Mermin operator is defined as

$$B_M = \hat{a}_1.\vec{\sigma} \otimes \hat{a}_2.\vec{\sigma} \otimes \hat{a}_3.\vec{\sigma} - \hat{a}_1.\vec{\sigma} \otimes \hat{b}_2.\vec{\sigma} \otimes \hat{b}_3.\vec{\sigma} - \hat{b}_1.\vec{\sigma} \otimes \hat{a}_2.\vec{\sigma} \otimes \hat{b}_3.\vec{\sigma} - \hat{b}_1.\vec{\sigma} \otimes \hat{b}_2.\vec{\sigma} \otimes \hat{a}_3.\vec{\sigma}$$

$\hat{a}_j, \hat{b}_j$ $(j = 1, 2, 3)$ are the measurement direction for the j^{th} party and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the usual Pauli matrices.

Ref.: N. D. Mermin, Phys. Rev. Lett.65, 1838 (1990);
D. P. Chi, K. Jeong, T. Kim, K. Lee, and S. Lee, Phys. Rev. A 81, 044302 (2010).

Procedure of choosing the operators

The operators can be obtained by suitably choosing the unit vectors in Mermin operator.

(i) The operator O can be constructed by choosing the unit vector as

$$\hat{a}_1 = \hat{a}_2 = \hat{a}_3 = (1,0,0), \hat{b}_1 = (-1,0,0), \hat{b}_2 = \hat{b}_3 = (1,0,0).$$

(ii) The operator O_1 can be constructed by choosing the unit vector as

$$\hat{a}_1 = \hat{a}_2 = (1,0,0),\ \hat{a}_3 = (0,0,1),\ \hat{b}_1 = (-1,0,0), \hat{b}_2 = (1,0,0), \hat{b}_3 = (0,0,1).$$

(iii) The operator O_2 can be constructed by choosing the unit vector as

$$\hat{a}_1 = \hat{a}_3 = (1,0,0),\ \hat{a}_2 = (0,0,1),\ \hat{b}_1 = (-1,0,0), \hat{b}_2 = (0,0,1), \hat{b}_3 = (1,0,0).$$

(iv) The operator O_3 can be constructed by choosing the unit vector as

$$\hat{a}_1 = (0,0,1),\ \hat{a}_2 = \hat{a}_3 = (1,0,0),\ \hat{b}_1 = (0,0,-1), \hat{b}_2 = \hat{b}_3 = (1,0,0).$$

Expectation values of operators

(i) The expectation value of the operator O in the state ψ is given by

$$\langle O\rangle_{\psi} = \langle\psi|O|\psi\rangle = 4\lambda_0\lambda_4 = 2\sqrt{\tau_{\psi}}$$

(ii) The expectation value of the operator O_1 in the state ψ is given by

$$\langle O_1\rangle_{\psi} = \langle\psi|O_1|\psi\rangle = 4\lambda_0\lambda_3$$

(iii) The expectation value of the operator O_2 in the state ψ is given by

$$\langle O_2\rangle_{\psi} = \langle\psi|O_2|\psi\rangle = 4\lambda_0\lambda_2$$

(iv) The expectation value of the operator O_3 in the state ψ is given by

$$\langle O_3\rangle_{\psi} = \langle\psi|O_3|\psi\rangle = -4(\lambda_2\lambda_3 + \lambda_1\lambda_4\cos\theta)$$

Two important quantities P and Q

Let us consider two quantities P and Q, which can be defined as

$$P = \langle O_1 \rangle_\psi \langle O_2 \rangle_\psi$$
$$= 16\lambda_0^2 \lambda_2 \lambda_3$$

$$Q = \langle O_1 \rangle_\psi + \langle O_2 \rangle_\psi + \langle O_3 \rangle_\psi$$
$$= 4(\lambda_0\lambda_3 + \lambda_0\lambda_2 - (\lambda_2\lambda_3 + \lambda_1\lambda_4 \cos\theta))$$

Classification of pure three qubit states: W class and Biseparability in 1 and 23 cut

(A) Any three-qubit pure state belongs to the W class if

(i) $\tau_{\psi} = 0$

(ii) $P \neq 0$

(b1) Any three-qubit pure state biseparable in 1 and 23 bipartition if

(i) $\tau_{\psi} = 0$

(ii) $\langle O_1 \rangle_{\psi} = 0$

(iii) $\langle O_2 \rangle_{\psi} = 0$ and

(iv) $\langle O_3 \rangle_{\psi} \neq 0$

Classification of pure three qubit states: Biseparability in different cuts

(b2) Any three-qubit state biseparable in 12 and 3 bipartition if

(i) $\tau_{\psi} = 0$

(ii) $\langle O_1 \rangle_{\psi} \neq 0$

(iii) $\langle O_2 \rangle_{\psi} = 0$ and

(iv) $\langle O_3 \rangle_{\psi} = 0$

(b3) Any three-qubit state biseparable in 13 and 2 bipartition if

(i) $\tau_{\psi} = 0$

(ii) $\langle O_1 \rangle_{\psi} = 0$

(iii) $\langle O_2 \rangle_{\psi} \neq 0$ and

(iv) $\langle O_3 \rangle_{\psi} = 0$

Classification of pure three qubit states: Biseparable and Separable

(B) Any three-qubit pure state is biseparable if

(i) $\tau_{\psi} = 0$

(ii) $P = 0$ and

(iii) $Q \neq 0$

(C) Any three-qubit pure state is separable if

(i) $\tau_{\psi} = 0$

(ii) $P = 0$ and

(iii) $Q = 0$

Local unitary equivalence with computational basis

Can we say that the above results are still valid, if we are given a state in any other basis instead of canonical form?

In principle one can always apply some local unitary operators to convert a state from any basis, in particular computational basis, to canonical-form and vice versa.

A. Acin, A. Andrianov, L. Costa, E. Jane, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).

We will now show that the previous stated results will hold in any basis with suitably transformed operators. We have to find the particular local unitary operation that connect two sets of basis vectors and write those operator in that basis.

Local unitary equivalence with computational basis: Example-1

Example-1: Let us consider a three-qubit pure state in a computational basis

$$|\psi\rangle_c = \frac{1}{2}(|000\rangle_c + |011\rangle_c + |100\rangle_c + |111\rangle_c)$$

Apply local unitary operators U_1, U_2, U_3 on first, second and third qubit, where

$$U_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; U_2 = U_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then by applying the operator $U = U_1 \otimes U_2 \otimes U_3$ on $|\psi\rangle_c$,

we get the state in the canonical form as

$$|\psi\rangle_a = \frac{1}{2}(|100\rangle_a + |111\rangle_a)$$

For the state $|\psi\rangle_a$, we find that $\langle O\rangle_{\psi_a} = 0, \langle O_1\rangle_{\psi_a} = 0, \langle O_2\rangle_{\psi_a} = 0, \langle O_3\rangle_{\psi_a} = -2$

Thus we can infer that the state is biseparable in 1 and 23 bipartition.

Local unitary equivalence with computational basis: Example-2

Example-2: Let us consider a three-qubit pure state in a computational basis

$$|\phi\rangle_c = \frac{1}{\sqrt{3}}(e^{i\theta}|000\rangle_c + |011\rangle_c - |100\rangle_c)$$

Apply local unitary operators U_1, U_2, U_3 on first, second and third qubit, where

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ U_2 = U_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then by applying the operator $U = U_1 \otimes U_2 \otimes U_3$ on $|\phi\rangle_c$,

we get the state in the canonical form as

$$|\phi\rangle_a = \frac{1}{\sqrt{3}}(|000\rangle_a + e^{i\theta}|100\rangle_a + |111\rangle_a)$$

For the state $|\phi\rangle_a$, we find that $\langle O\rangle_{\phi_a} = \frac{4}{3}, \langle O_1\rangle_{\phi_a} = 0, \langle O_2\rangle_{\phi_a} = 0, \langle O_3\rangle_{\phi_a} = \frac{-4\cos\theta}{3}$

Thus we can infer that the state is in GHZ class of states.

Tangle for three qubit mixed states

Recapitulation:

Tangle for three qubit pure state is given by

$$\tau = C^2_{A(BC)} - C^2_{AB} - C^2_{AC}$$

where, $C^2_{A(BC)} = 2(1 - Tr(\rho_A^2))$

But there is no closed form of tangle for mixed state.

For three qubit mixed state

$$C^2_{A(BC)}(\rho) = \inf_{p_i,\psi_i} \sum_i p_i C^2_{A(BC)}(|\psi_i\rangle)$$

where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Lower bound of tangle of mixed state

Lower bound of $C^2_{A(BC)}(\rho)$:

$$C^2_{A(BC)}(\rho)|_{LB} = 2(Tr(\rho^2) - Tr(\rho_A^2))$$

Let $\tau^{LB}(\rho)$ denote the lower bound on tangle.

$\tau^{LB}(\rho)$ is not always invariant under the permutation of A, B and C.

So it is reasonable to use the average over all the permutations of A, B and C.

Average value of lower bound of tangle is given by

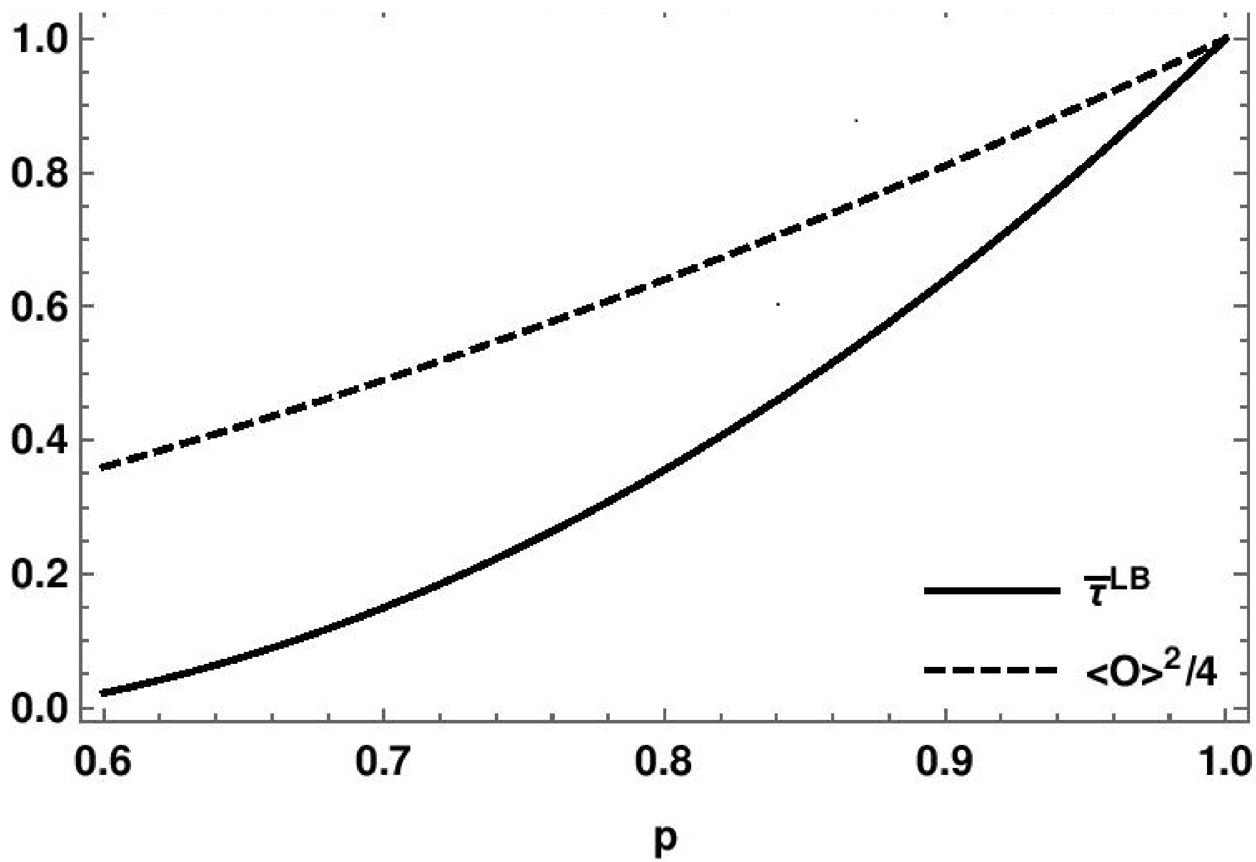
$$\overline{\tau}^{LB} = \frac{1}{6}\sum_{\{A,B,C\}}(C^2_{A(BC)}|_{LB} - C^2_{AB} - C^2_{AC})$$

F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 98, 140505 (2007).

Example-1

Consider a mixture of GHZ and W state :

$$\rho = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|$$



Example-2

Prepare a two-qubit Bell state: $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$

Phase damping channel is described by the transformation

$$|0\rangle_B |0\rangle_E \rightarrow |0\rangle_B |0\rangle_E$$

$$|1\rangle_B |0\rangle_E \rightarrow \sqrt{1-p}\,|1\rangle_B |0\rangle_E + \sqrt{p}\,|0\rangle_B |1\rangle_E$$

where p is the channel parameter.

A three-qubit state generated is given by

$$|\phi\rangle_{ABE} = \frac{1}{\sqrt{2}}(|000\rangle + \sqrt{1-p}\,|110\rangle + \sqrt{p}\,|111\rangle)$$

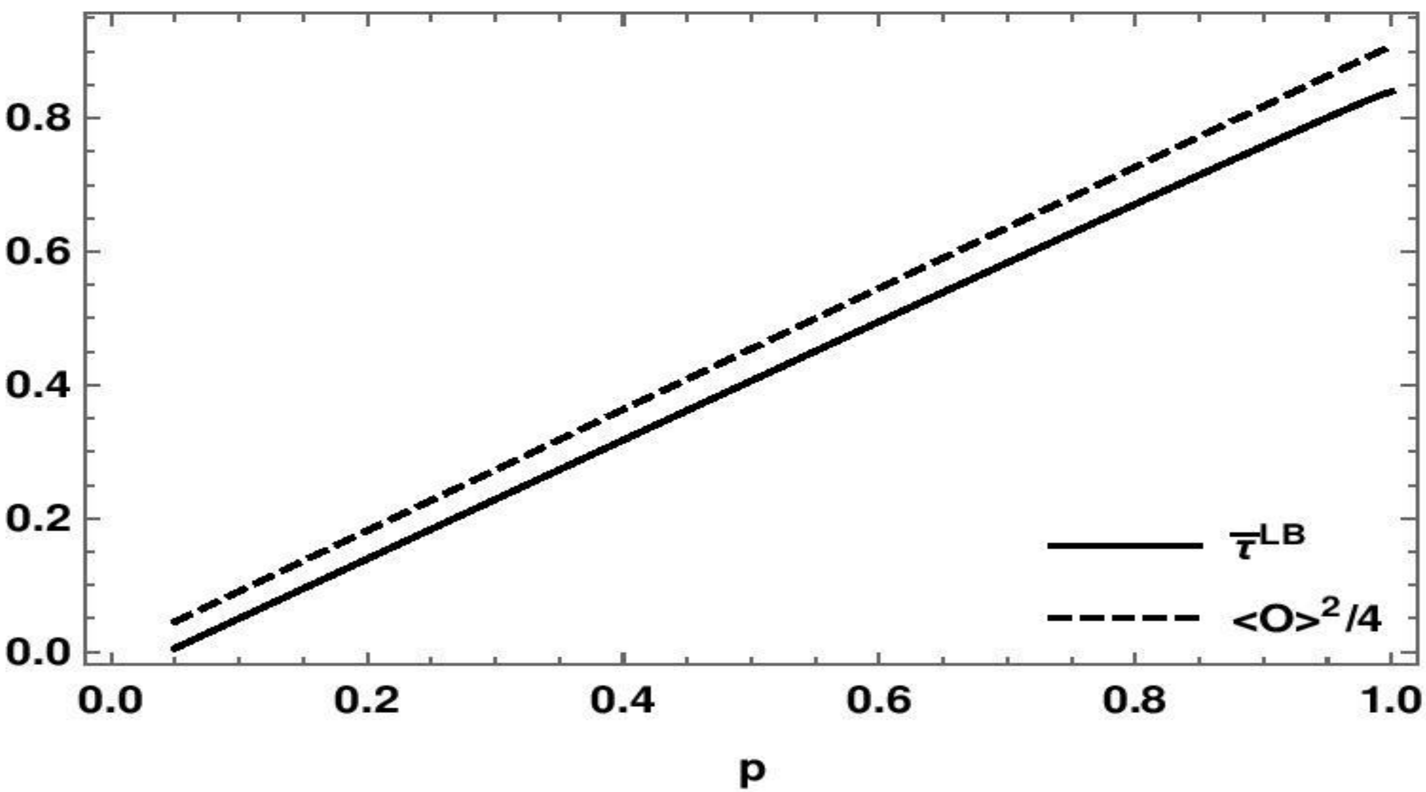
Consider a mixture of $|\phi\rangle_{ABE}\langle\phi|$ and white noise I :

$$\rho = m|\phi\rangle_{ABE}\langle\phi| + \frac{1-m}{8} I$$

Example-2(Continued)

Mixture of $|\phi\rangle_{ABE}\langle\phi|$ and white noise I: $\rho = m|\phi\rangle_{ABE}\langle\phi| + \frac{1-m}{8}I$

Compare numerically our tangle measure for pure state with the lower bound of tangle.



Application

Let us consider a three-qubit pure entangled state shared by three parties i, j and k.

We make an orthogonal measurement on the kth qubit and consider the joint state of the system i and j.

Using this joint state as a resource state, one can teleport a single qubit state.

The faithfulness of this teleportation scheme depends on the single qubit measurement on kth qubit and the compound state of the system i and j.

Ref: S. Lee, J. Joo, and J. Kim, Phys. Rev. A 72, 024302 (2005).

Partial Tangle

Partial tangle is defined as

$$\tau_{ij} = \sqrt{C_{i(jk)}^2 - C_{ik}^2},\ i \neq j \neq k\ \ and\ \ i, j, k = 1, 2, 3$$

The partial tangle for the state $|\psi\rangle_{123} = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$,
$\lambda_i \geq 0, \quad \sum \lambda_i^2 = 1, \quad \theta \in [0, \pi]$ is given by

$$\tau_{12} = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2},\ \tau_{31} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_4^2},\ \tau_{23} = 2\sqrt{\lambda_0^2\lambda_4^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4 cos\theta}$$

Relation between the partial tangle τ and maximum teleportation fidelity F:

$$\tau_{ij} = 3F_k - 2$$

Ref: S. Lee, J. Joo, and J. Kim, Phys. Rev. A 72, 024302 (2005).

Relation between partial tangle, teleportation fidelity and expectation of operators

Maximum teleportation fidelity in terms of the expectation value of the operators:

$$\tau_{12} = \frac{1}{2}\sqrt{\langle O\rangle_\psi^2 + \langle O_1\rangle_\psi^2} = 3F_3 - 2,$$

$$\tau_{23} = \frac{1}{2}\sqrt{\langle O\rangle_\psi^2 + \langle O_4\rangle_\psi^2 + \langle O_5\rangle_\psi^2} = 3F_1 - 2$$

$$\tau_{31} = \frac{1}{2}\sqrt{\langle O\rangle_\psi^2 + \langle O_2\rangle_\psi^2} = 3F_2 - 2,$$

where $O_4 = 2(\sigma_z \otimes \sigma_y \otimes \sigma_y), O_5 = 2(\sigma_z \otimes \sigma_y \otimes \sigma_x)$

Results on the partial tangle and separability

Result−1: If all the partial tangles are equal to zero then the state is a separable one. Because the expectation value of O_4 and O_5 are also zero for a separable state.

Result−2: If at least one partial tangle is equal to zero, then the three−qubit state is a biseparable state..

Result−3: If each partial tangle is not equal to zero then the state is a three−qubit genuine entangled state.

Result−4: Any pure three−qubit genuinely entangled state is useful in the teleportation scheme given by Lee et.al.

S. Lee, J. Joo, and J. Kim, Phys. Rev. A 72, 024302 (2005).

Experimental Implementation

We undertake experimental detection of the entanglement present in arbitrary three-qubit pure quantum states on an NMR quantum information processor. Measurements of only four observables suffice to experimentally differentiate between the six classes of states which are in equivalent under stochastic local operation and classical communication. The experimental realization is achieved by mapping the desired observables onto Pauli z operators of a single qubit, which is directly amenable to measurement. The detection scheme is applied to known entangled states as well as to states randomly generated using a generic scheme that can construct all possible three-qubit states. The results are substantiated via direct full quantum state tomography as well as via negativity calculations and the comparison suggests that the protocol is indeed successful in detecting tripartite entanglement without requiring any a priori information about the states.

C. Dutta, Satyabrata Adhikari, A. Das, P. Agrawal, Euro. Phys. J. D 72, 157 (2018)
A. Singh, H. Singh, K. Dorai, and Arvind, Phys. Rev. A 98, 032301 (2018).

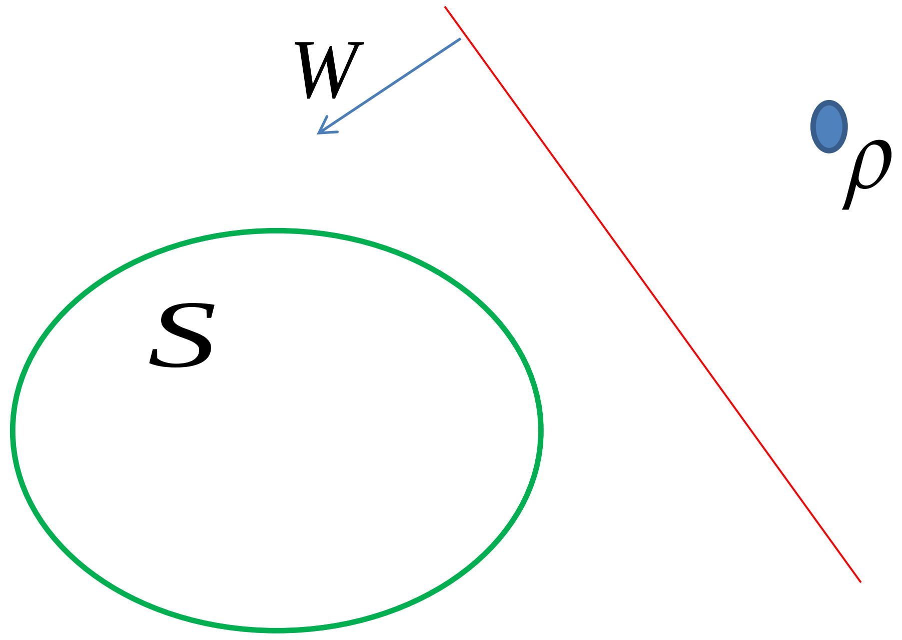
NOTE: Teleportation fidelities may be measured experimentally.

Detection of Entangled States

Generation and detection of entanglement

Experimentally: The entanglement generated in a laboratory is never free from imperfections and noise. Therefore, they may but do not have to be entangled. For this reason, it is important to develop efficient and easy to apply experimental procedure to detect entanglement.

Theorem : Let S be a convex and compact set in a finite dimensional Banach space. Let ρ be a point in the space with $\rho \notin S$. Then there exists a hyperplane that separates ρ from S.



Entanglement witness

- *An entanglement witness operator W is a Hermitian operator which satisfies the following inequalities*

$$(i)\ Tr(W\sigma) \geq 0,\ \text{for all separable}\ \sigma$$

$$(ii)\ Tr(W\rho) < 0,\ \text{for at least one entangled}\ \rho$$

Properties of Entanglement Witnesses (EW)

- EW have negative eigenvalues.

- *Since EW are hermitian, they can be treated as physical observables and hence they are interesting from the experimental point of view.*

- *EW depend on the states in the sense that there exist entangled states that are only detected by different witnesses.*

Detection of Entanglement

- The construction of a witness operator for a given arbitrary state is, in general, a difficult task.
- However, one can construct a witness operator if one has some a priori information about the density matrix. This is always the case when the experiment is aimed at producing a certain state.

Witness operator

The three-qubit mixed entanglement in different classes were characterized using witness operators.

Example:

$$W_{GHZ} = \frac{3}{4} I - P_{GHZ}, \ where \ P_{GHZ} = |GHZ><GHZ|, |GHZ> = \frac{1}{\sqrt{2}}(|000> + |111>)$$

This witness operator is useful to distinguish three qubit states in GHZ or *W classes.*

But it fails to detect whether the state

$$\left|\chi^{(1)}\right\rangle = \frac{1}{\sqrt{2}}[\sin\theta\left|000\right\rangle + \sin\theta\left|011\right\rangle + \cos\theta\left|110\right\rangle - \cos\theta\left|101\right\rangle], \ \ 0 \le \theta \le \frac{\pi}{4}$$

belongs to GHZ or *W class.*

Construction of witness operator

Let us consider the witness operator of the form

$$W_\chi = \varepsilon I - P_\chi,\ P_\chi = |\chi\rangle\langle\chi|,\ |\chi\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |011\rangle + |110\rangle + |101\rangle]$$

Where ε is the maximal overlap between the states

$|\chi^{(1)}\rangle$ and $|W\rangle = r_1|000\rangle + r_2[|001\rangle + |010\rangle + |100\rangle]$

Performing the optimization procedure, we get $\varepsilon = \frac{1}{4}$.

Therefore, we the witness operator takes the form

$$W_\chi = \frac{1}{4} I - P_\chi$$

Decomposition of witness operator

It can be easily shown that

$$(i)\ \ Tr\ (W_\chi P_W) \geq 0, \text{ for all } |W\rangle$$

$$(ii)\ Tr\ (W_\chi P_{\chi^{(1)}}) = -(1 + 2Sin2\theta)\ < 0, \text{ for } 0 \leq \theta \leq \frac{\pi}{4}.$$

This witness operator detects all the states in $|\chi^{(1)}\rangle$

and shows that the states belong to the GHZ *classes.*

The witness operator W_χ Can be decomposed as

$$W_\chi = \frac{1}{8}[I \otimes I \otimes I + \sigma_y \otimes \sigma_y \otimes I - \sigma_y \otimes I \otimes \sigma_y + I \otimes \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_x \otimes \sigma_x$$
$$+ \sigma_x \otimes \sigma_z \otimes \sigma_x - \sigma_x \otimes \sigma_x \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes \sigma_z]$$