Quantum Entanglement

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Quantum information methods are now finding application in a wide variety of areas. In particular “Quantum entanglement” seems to be playing role not only in different branches of physics - condensed matter physics, high energy physics - and computation, but also in chemistry and biology. Entanglement seems to be ubiquitous. Apart from being behind some of the counter-intuitive aspects of quantum mechanics, it is also turning out to be a useful resource. But Beyond the simplest system of two qubits in pure state, one still does not know how to fully characterize and quantify the entanglement in more complicated systems. To make efficient use of entanglement, one has to understand it better in more realistic systems. we will look at some aspects of entanglement in simple contexts.
Introduction

Why some finds aspects of quantum mechanics mysterious? What makes quantum systems such powerful resources?

I think I can safely say that nobody understands quantum mechanics.
Character of Physical Law, 1964

We always have had a great deal of difficulty in understanding the world view that quantum mechanics represents. At least I do, because I’m an old enough man that I haven’t got to the point that this stuff is obvious to me. Okay, I still get nervous with it.
Simulating Physics with Computers, 1982
Introduction

For those who are not shocked when they first come across quantum theory cannot possibly have understood it.

Letter to W. Heisenberg

God does not play dice.

Letter to M. Born
Introduction

Nonlocality is unavoidable, even if it looks like "action at a distance".
A talk at CERN, 1990

There is a troubling weirdness about quantum mechanics. Perhaps its weirdest feature is entanglement, the need to describe even systems that extend over macroscopic distances in ways that are inconsistent with classical ideas.
Lectures on Quantum Mechanics, 2015
Introduction

- Even when some of the pioneers differed about the interpretation and the world-view of quantum mechanics formalism, they agree that there is some mysterious aspects that a quantum system has.
- Irrespective of point of view you take, one has come to believe that there are things that one can do using quantum mechanical resources, that are not possible using classical resources.
- It is one of the source of quantum mechanics mysterious aspects that give QM resources more power.
- What is it that is special about QM resources?
- Answer is **Superposition**.
- If we consider a two-state system, if it is classical, we can characterize the states with, say $|0\rangle$ and $|1\rangle$. Then the system can be only in one of these states.
- OTOH, a quantum system can exist in a superposition of these two states also

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- If we make a measurement to find the state of the system, It will be found in $|0\rangle$ with probability $|\alpha|^2$ and in $|1\rangle$ with probability $|\beta|^2$. 
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- Let us now consider composite systems which consist of many parts. We can then write the state of such a system in terms of the possible states of the subsystems. For example, for a two-qubit system, the state can be a product state

$$|\psi\rangle_{AB} = |0\rangle_A |1\rangle_B$$

- This composite system can be in many such states, or their superposition, e.g.,

$$|\psi\rangle_{AB} = \alpha |0\rangle_A |1\rangle_B + \beta |1\rangle_A |0\rangle_B$$

- These states are called entangled states and can not be written as product states.

- By definition, a pure state is said to be an entangled state if it cannot be written as a product state of subsystems. That is,

$$|\psi\rangle_{AB} \neq |\chi\rangle_A |\eta\rangle_B$$

Otherwise it is called an entangled state.
Quantum Entanglement

- It is clear that an entangled state is a superposed state. But all states which appear to be superposed, may not be entangled. If we choose an appropriate basis, it would be a product state – so not entangled.

- A very good example of such a state is:

\[ |\psi\rangle_{AB} = \frac{1}{2} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B + |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \]

This state appears to be entangled – may be even “maximally” entangled. But it is not entangled. We can rewrite this state in another basis as

\[ |\psi\rangle_{AB} = |+\rangle_A |+\rangle_B \]

- Most famous two-qubit entangled states are Bell states. These are:

\[ |\psi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B) \]

\[ |\phi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B \pm |1\rangle_A |1\rangle_B) \]

As we will see later, in some sense, these states are maximally entangled. But how do we know that these states are even entangled? We have seen that appearance can be deceptive.
Quantum Entanglement

• For a pure two-qubit state it is easy to check. All one has to do is to check that this state cannot be written as a product of general two one-qubit states – $\alpha_1 |0\rangle + \alpha_2 |1\rangle$ and $\alpha_3 |0\rangle + \alpha_4 |1\rangle$.

• Exercise - Check that Bell states are indeed entangled states.

• Once we go beyond two-qubit states to multi-qubit states, or more complicated multi-qudit states, the procedure becomes more cumbersome. But, there exist ways to be sure.

• As we will see later, if we go beyond pure state, there are cases where there does not exist a method to be sure that a given state is entangled.

• These are the examples of entangled pure state. A quantum system can be in a more complicated state – a mixed state.

• To describe these states we need a description in terms of density matrices.
Quantum Entanglement

- A density matrix of a system can be written as
  \[ \rho = \sum_j p_j |\varphi_j\rangle \langle \varphi_j| \]

  Here \( p_j \) is the probability of the system being in the state \( |\varphi_j\rangle \).

- This operator has many properties. Its trace is one; it is hermitian; it is positive, so all its eigenvalues are greater or equal to zero.

- For a pure state, only one of the \( p_j \) is nonzero.

- A density matrix can have many different decompositions. This will give rise to complications when we have to characterize its entanglement.

- For example, for a one-qubit system
  \[ \rho = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|) \]
  \[ = \frac{1}{2} (|+\rangle \langle +| + |--\rangle \langle --|) \]

- Exercise: Find another decomposition of the density matrix
  \[ \rho = \frac{3}{4} |0\rangle \langle 0| + \frac{1}{4} |1\rangle \langle 1| \]
Quantum Entanglement

• We can now define the mixed states that are entangled. A density matrix of a bipartite system is entangled, if it cannot be written as a mixture of product states. If it can be, then we say that the state is separable.
• So, in the case of mixed states, a separable state can be written as

\[ \rho^{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B \]

i.e., a mixture of product states.
• Just like for pure states, by just looking at a state, one cannot easily say if the state is entangled or not. In fact, the procedure for a mixed can be more complicated.
• Let us look at some examples.

\[ \rho_{AB}^1 = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|) \]

\[ \rho_{AB}^2 = \frac{1}{2}(|++\rangle \langle ++| + |--\rangle \langle --|) \]

These density matrices look different, but they describe the same system state. They are just different decompositions.
Quantum Entanglement

- Let us now look at a mixture of entangled states
  \[ \rho_{AB}^3 = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|) \]

- Naively, one may think that it is a mixture of entangled state, so it may be entangled.
- Exercise: Show that this density matrix can be written as a mixture of product states.
- Given a density matrix, it is non-trivial to find out, if it describes an entangled state or not. Even in the case of as simple a system as two-qutrits, we may miss entangled states, whatever criteria, we may use.
- One idea here is to check if a Bell inequality is violated. It turns out that this idea does not always work.
- However for two-qubit mixed states, we do have a necessary and sufficient criterion for this purpose. It is called Peres-Horodecki criterion.
- In this criterion, one uses a positive map, transposition operator: \( \rho \rightarrow \rho^T \). A state is entangled iff
  \[ \rho^T_{AB} = (I \otimes T^B)\rho_{AB} \]
is not positive. This is partial transpose. It means take transposition for system B, leaving system A alone.
Quantum Entanglement

- Let us check if a Bell state is entangled by this criterion or not.
- Let us consider the Bell state $|\phi^+\rangle$. In computational basis, the density matrix can be written as

$$\rho_\phi = \frac{1}{2} (|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|)$$

- After partial transpose:

$$\rho^{T_B}_\phi = \frac{1}{2} (|00\rangle \langle 00| + |01\rangle \langle 10| + |10\rangle \langle 01| + |11\rangle \langle 11|)$$

- $|\phi^+\rangle$ is entangled if $\rho^{T_B}_\phi$ has a negative eigenvalue. In computational basis, the partial transpose matrix is

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

- You can check that it has a negative eigenvalue $-\frac{1}{2}$. So the state is entangled as it must.
A famous two-qubit mixed state is Werner state:

\[ \rho^w = p|\phi^+\rangle\langle\phi^+| + \frac{(1-p)}{4} I \]

Here \( I \) is identity operator, which can be written in terms of all Bell states.

• This state is quite intriguing. It is entangled for \( p > \frac{1}{3} \).

• This state shows a phenomenon of ”hidden nonlocality”. Even in some of the \( p \) region where it is entangled, the state does not violate Bell inequality.

• Many other mixed states also show similar feature.

• **Exercise**: Check that the Werner state is entangled if \( p > \frac{1}{3} \).
EPR Paradox

- It is superposition and its manifestation as entanglement for composite system, which give rise to many enduring mysteries about quantum mechanics. EPR paradox (Einstein-Podolsky-Rosen, 1935) illustrates this quite well.
- The paper introduced the notion of realism as:
  
  \textit{If, without anyway disturbing a system, we can predict with certainty, the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.}

- The notion of locality was introduced as:

  \textit{Elements of physical reality belonging to one system cannot be affected by measurement performed on another system which is spatially separated from the former.}

- Then there was a condition for a theory to be complete:

  \textit{A necessary condition for a theory to be complete is that every element of physical reality must have a counterpart in the theory.}
• EPR then analysed a particular state of two particles involving position and momentum in one dimension and argued for the incompleteness of the quantum theory.

• There is Bohm version of this analysis. Bohm considered two spin-half particles in the singlet state (|φ−⟩). An analysis suggests that each particle should have simultaneous definite values of spin components in say x and z directions. Quantum mechanics precludes this, thus making it incomplete.

• To make the QM complete, the idea of hidden-variables was introduced. It was shown by Bell that local hidden-variable models are not compatible with QM. Quantum mechanics does violate EPR notion of local-realism.
EPR Paradox

- EPR paper prompted Schrodinger to introduced two terms – a) *entangled state* (for a state like singlet state) and b) *steering* (ability of each party to steer the state of the other party by making a measurement)
- Theses notions remain dormant for more than 25 years until Bell resurrected them by introducing his inequality and showing its violation by QM.
Quantum Cryptography

- In this school/workshop you may have heard about the famous BB84 quantum cryptography protocol (Bennett-Brassard, 1984). The protocol involves superposed states and transmitting quantum systems. This protocol did not require entanglement. One does measurement in computational or Hadamard basis.

- It was Ekert in 1991, as you have already heard from many speakers who used entangled states for Alice and Bob to establish a secret key. In the original version, Alice and Bob shared many copies of a Bell state.

- So it is the ability of a quantum system to be in a superposed state that is making quantum systems more powerful resources.
Quantum Communication

- There are of course many tasks that one can carry out with the assistance of entanglement. Some of these tasks are only possible with quantum systems.
- We have mentioned Bell inequality violation and quantum cryptography protocols.
- Study of entanglement took off with the introduction of many communication protocols and possibility of more powerful algorithms for some computing tasks.
- In communication protocols, some early ones that you may have seen are teleportation, superdense coding, entanglement swapping, secret sharing etc. In most of these protocols one party, say Alice, can communicate quantum states, classical information, to Bob either directly, or with the help of one or more parties.
Quantum Entanglement

- Once we realize that entangle is important, the question can be asked how entangled a state is?
- Which entangled state is more useful for a specific task?
- Can we say that a state is more entangled than the other?
- Are more entangled states always more useful?
- Or even more elementary question - is the given quantum state entangled?
- Answer to these and many more questions depend on many things. Some of these things are: Is the system in pure state or mixed state? Is the system bipartite, or multi-partite? Do we have qubits, or higher dimensional systems?
- Some of these questions, we have already discussed.
Entanglement Measure

- We will now discuss how to quantify entanglement for a two-qubit system. We will first discuss the case of a pure state.
- Most popular measure of entanglement of such a system is von Neumann entropy of the reduced density matrix of one of the subsystems. If the state is $|\psi\rangle_{AB}$ then this measure is:

$$E(|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)$$

- Von Neumann entropy is

$$S(\rho) = -\text{Tr}(\rho \log_2 \rho) = -\sum_{\ell} \lambda_\ell \log_2 \lambda_\ell$$

- For a product state von Neumann entropy is zero. For Bell states it is one.
- Exercise: Compute these entropies.
- Von Neumann entropy satisfy some of the desirable property for a measure - additivity for independent systems, invariance under local unitary transformations, and on average its value does not increase under LOCC.
Entanglement Measure

- There are many other measures, viz, concurrence, distillable entanglement, entanglement of formation, negativity, relative entropy, etc.
- But we will discuss only one more - Concurrence.
- One defines a tilde state as

\[ |\tilde{\psi}\rangle = (\sigma_2 \otimes \sigma_2)|\psi^*\rangle \]

Here complex conjugation is in computational basis.
- Then, concurrence is defined as:

\[ C(|\psi\rangle) = |\langle \psi |\tilde{\psi}\rangle| \]

- For a product state it is zero, while it is one for a Bell state. Another measure, entanglement of formation can be expressed in its terms.
- Importance of this measure comes from easy computability and also to obtain an expression for a mixed state.
- For a pure state in Schmidt decomposition, concurrence is \( 2\sqrt{\lambda_1\lambda_2} \).
Entanglement Measure

- For a mixed state, a complication arises due to multiple decompositions of a density matrix.
- For the density matrix
  \[ \rho_{AB} = \sum_i p_i \rho_{AB}^i \]
  one defines the measure as
  \[ E(\rho_{AB}) = \min \sum_i p_i E(\rho_{AB}^i) \]
  Here one minimizes over all possible pure state decompositions of \( \rho_{AB} \).
- Such minimizations are hard. Wooters has done it for concurrence. He has shown that
  \[ C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \]
  Here \( \lambda_i \) are square root of eigenvalues of \( \tilde{\rho} \) where
  \[ \tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho^* (\sigma_2 \otimes \sigma_2) \]
Multipartite Entanglement

- As we move beyond bipartite systems, the story becomes more complex. A multipartite system has many particles. It can have many different subsystems. These subsystems can entangle in different ways, making it difficult for some kind of universal characterization.
- For such systems, there is also an issue of genuine multipartite entanglement. Sometimes, an entangled subsystem may not be entangled with other subsystems. For example, the state
  \[ |\psi\rangle_{ABCD} = |\phi^+\rangle_A |\psi^-\rangle_{CD} \]
does not have genuine multipartite entanglement.
- Actually, one of the important problems in the field of Quantum Information is to understand the nature of entanglement in multipartite systems.
- One can do a lot more when there are many particles and the nature of entanglement is more complex.
- Such systems exhibit some intriguing phenomena which don’t occur for a bipartite (specially qubit) system.
Multipartite Entanglement

- This is a very rich field. We will touch upon three different phenomena associated with multipartite systems.
- Is there a meaningful notion of maximally entangle state? For the multipartite states, it may depend on what we wish to do. It is task dependent. In this context, we will discuss the notions of “Task-oriented Maximally Entangled State” (TMES), and “Absolutely Maximally Entangled State (AMES)”.
- In the case of multipartite system, in some situations one can do more communication with a less entangled state. We consider teleportation, superdense coding, and Quantum Key distribution to illustrate this feature of “Less is More”.
- This can be understood due to the need for special amount of entanglement for a specific task. We will discuss a condition for the teleportation of the state of a $n$-qubit system with $m$ terms.
- A third aspect that we will touch upon is “Monogamy of Entanglement”. This basically tells us that, for example, in the case of a tripartite system, entanglement between A and B limits their entanglement with C.
Maximally Entangled States

- As we saw, for a pure bipartite state, the entanglement can be characterized and quantified by the von Neumann entropy (or some other equivalent measure). This measure maps a bipartite state to a real number. One can order the states according to von Neumann entropy.
- Because of the existence of a suitable entanglement measure for a bipartite system, there exists the concept of a maximally entangled state. Von Neumann entropy is zero for the unentangled systems and one for the maximally entangled systems. The states of the maximally entangled bipartite systems are Bell states.
- Beyond a bipartite system, there is no consensus about what states might be considered maximally entangled. This is because we don’t have a suitable measure.
- In the case of a multipartite entangled state, there exist multiple bipartite, tripartite and higher entanglement. Therefore, one number may not be suitable to characterize such states. One may need a set of numbers to characterize such states.
Maximally Entangled States

- In the case of bipartite pure entangled states, the notion of maximally entangled state is unambiguous. For a two-qubit system, these are well-known Bell states. By using LOCC, any other two-qubit state can be obtained from these states.

- As we go beyond bipartite case, the situation is not clear.

- For multipartite case, multiple notions exist. Two of these are concepts of Absolutely Maximally Entangled State (AMES) and Task-Oriented Maximally Entangled States (TMES).

- If a system is in AMES, then all its subsystems will have maximally allowed entropy. For the case of three qubits, GHZ state is AMES. For a four-qubit system, there are no AMES. There exist AMES for five and six-qubit systems. For eight-qubit systems and beyond there are no AMES. For a seven-qubit also there appears to be no AMES. In all such cases, one may look for states close to AMES.
Task-oriented Maximally Entangled State

- There does not exist an analog of Bell states for multipartite systems and this notion may not be universal. There may exist a global maximum relative to a specific entanglement measure, but the state with such a property may not be most suitable for most tasks that one may envision.
- This brings us to the notion of task-oriented maximally entangled state (TMES). TMESs are those states that may be suitable to carry out a specific task \textit{maximally}. For different tasks, one may need different TMESs.
- A task can be Bell inequalities or some equalities which uses the entanglement of a quantum state or some communication or processing of information. Some of the communication based tasks are teleportation, superdense coding, multi-receiver superdense coding, quantum cryptography, secret sharing, telecloning, etc.
- We will discuss TMES for the tasks of simple teleportation and superdense coding.
Maximally Entangled States

- As we will see later, in the case of three-qubit case, the GHZ state is TMES for a number of tasks, like teleportation, superdense coding, secret sharing, etc.
- For a three-qubit system, a modified-W state that is suitable for perfect teleportation and superdense coding can also be a TMES. But for a given modified-W state, it is possible in only one specific partition, while one can use GHZ state in any partition.
- If we consider GHZ state for a three-qubit system to be maximally entangled, then one may expect it to be maximally nonlocal also. For a quantum-mechanical system, entanglement is the source of nonlocality.
- If we use violation of Bell inequality as a measure of nonlocality, then as we will see, the situation is not straight forward.
Monogamy of Entanglement

- In the case of multipartite systems, the entanglement between different subsystems cannot take arbitrary values. For example, consider a system of three qubits. If qubits A and B are maximally entangled, then they cannot entangle with a third qubit C. So, we will have a direct product state.

\[ |\psi\rangle_{ABC} = |\phi^+\rangle_{AB}|\eta\rangle_C \]

- There is a trade-off of entanglement between various subsystems.
- This notion was formalized by Coffman-Wooters-Kundu, using the Concurrence measure. They showed

\[ C^2_{A|B} + C^2_{A|C} \leq C^2_{A|BC} \]

- This inequality has been extended to n-qubit system also.
- Not all measures of entanglement exhibit monogamy. However, for various measure, specific classes of states may show monogamy.
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Quantum Nonlocality

- One of the features of QM that has puzzled some for a long time is violation of local-realism. This has led to a belief that QM exhibit nonlocality.
- As discussed earlier, it started with EPR paradox. It was given a more concrete shape with the introduction of Bell inequality to test local-realism in an experiment.
- Only recently, some loophole free experiments have been conducted to establish that quantum mechanical correlations indeed violate local-realism.
- We will briefly outline CHSH-Bell inequality, facet inequalities, and Hardy nonlocality argument in the context of a two-qubit system.
- Cases of bipartite qudits system and multipartite systems are more complicated.
- We will briefly discuss some famous and some recent Bell type inequalities in the context of multipartite systems.
Quantum Nonlocality

- In 1964, Bell obtained an inequality and demonstrated the incompatibility between local-realism and quantum mechanics. It was done for a singlet state.
- After more than 25 years, in 1991, Gisin showed that any pure entangled state of a bipartite system violates a Bell’s inequality, more accurately Clauser-Horne-Shimony-Holt (CHSH) inequality.
- Inequalities can be written in terms of correlations of observables. Maximum violation of CHSH inequality in quantum mechanics can be $2\sqrt{2}$ (Tsirelson’s bound, 1980).
- One can establish a relationship between entanglement and nonlocality in the case of pure bipartite states.
- The situation about mixed state is not clear. There are entangled states which don’t violate standard CHSH inequality. Prototype example is Werner state. There appears to exist the phenomenon of hidden nonlocality.
- The phenomena of hidden nonlocality is more of a norm, than exception. Multipartite mixed states also exhibit this phenomena.
- Case of a multipartite state is more complex because we don’t know how to characterize its entanglement. There can be multiple ways to characterize its nonlocality. But one may be able to discuss some categories of such states.
The CHSH inequality (John Clauser, Michael Horne, Abner Shimony, and Richard Holt, 1969) is given in terms of the following combination of the observables,

\[ I_{CHSH} = A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 \]

In a local-realistic theory,

\[ \langle I_{CHSH} \rangle \leq 2 \]
CHSH Inequality

- Let us now consider the measurement settings:

\[ A_1 = \sigma_x, \quad A_2 = \sigma_y, \]
\[ B_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \quad B_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y). \]

- We take the nonmaximally entangled state as,

\[ |\psi_0\rangle = c_0|00\rangle + c_1|11\rangle. \]

- Note that the state is in \( \sigma_z \) basis, while \( A_1 \) and \( A_2 \) are in the perpendicular directions.

- Then, we find,

\[ \langle \psi_0 | I_{CHSH} | \psi_0 \rangle = 4\sqrt{2}c_0c_1 = 2\sqrt{2}C. \]

Here \( C \) is the concurrence of the state.
CHSH Inequality

- Let us now consider another set of measurement settings:

\[
A_1 = \sigma_z, \quad A_2 = \sigma_x, \\
B_1 = \cos(\eta) \sigma_z + \sin(\eta) \sigma_x, \quad B_2 = \cos(\eta) \sigma_z - \sin(\eta) \sigma_x.
\]  

Here \( \cos(\eta) = \frac{1}{\sqrt{1+C^2}} \).

- For a nonmaximally entangled state

\[
|\psi_0\rangle = c_0|00\rangle + c_1|11\rangle
\]

we find,

\[
\langle\psi_0|l_{CHSH}|\psi_0\rangle = 2\sqrt{1+C^2}.
\]

Here \( C \) is the concurrence of the state.

- These settings are the best. For any entangled state there is a violation of CHSH inequality. These settings have been optimized for each state to give maximum possible value. (Horodecki\(^3\), 1995)
The Hardy’s nonlocality argument

- As before, consider a physical system consisting of two subsystems shared between Alice and Bob. The two observers (Alice and Bob) have access to one subsystem each.
- Assume that Alice can run the experiments of measuring any one (chosen at random) of the two \{+1, −1\}-valued random variables \( A \) and \( A' \) corresponding to her subsystem; similarly for Bob.
- Consider now the following four conditions:

\[
\begin{align*}
\text{Prob}(+1, +1|A, B, P) &> 0, \\
\text{Prob}(-1, +1|A', B, P) &= 0, \\
\text{Prob}(+1, -1|A, B', P) &= 0, \\
\text{Prob}(+1, +1|A', B', P) &= 0. \\
\end{align*}
\]

- The above four conditions together form the basis of Hardy’s nonlocality argument.
- The first condition says that in an experiment in which Alice chooses to measure the observable \( A \) and Bob chooses the observable \( B \), the probability that both will get +1 as measurement outcomes is nonzero. Other conditions can be analysed similarly.
The Hardy’s nonlocality argument

- The Hardy’s nonlocality argument makes use of the fact that these four conditions cannot be fulfilled simultaneously in the framework of a local-realistic theory.
- But they can be in quantum mechanics.
- In the framework of a local-realistic theory,

\[
\text{Prob}(a, b|A, B, P) = \int_{\Lambda} p(a|A, P, \lambda) \ p(b|B, P, \lambda)p(\lambda|P) \ d\lambda.
\]

- The first Hardy’s condition implies – \( p(+1|A, \lambda) > 0 \) and \( p(+1|B, \lambda) > 0 \).
- The second Hardy’s condition implies, \( p(-1|A^{'P}, \lambda) = 0 \), or equivalently for these \( \lambda \)s, \( p(+1|A^{'P}, \lambda) = 1 \).
- The third Hardy’s condition implies, \( p(-1|B^{'P}, \lambda) = 0 \), or equivalently for these \( \lambda \)s, \( p(+1|B^{'P}, \lambda) = 1 \).
- These three conditions taken together contradict fourth condition. This is because these conditions imply, \( \text{Prob}(+1, +1|A^{'}, B^{'}, P) > 0 \).
- Any non-maximally entangled two-qubit pure state satisfy Hardy’s four conditions.
Facet Inequalities

• One way to obtain inequalities is find facets of a polytope in the joint probability space.
• The CHSH inequality is a facet inequality. It is violated maximally by the maximally entangled two-qubit states, the Bell states.
• It was then the naive expectation that the same may happen for other systems.
• It was thus a surprise when CGLMP inequality (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002), which is a facet inequality for a two-qudit system was shown to be violated more by a partially entangled state. It was termed as “anomaly of nonlocality”. This inequality is based on $(2, 2, d)$ scenario. However, there are inequalities, like SLK (Son-Lee-Kim) inequalities, that are violated maximally by a maximally entangled two-qudit state. But these inequalities are not facet inequalities.
• As a illustration, we will first discuss CHSH inequality as a facet inequality. We will then consider three-qubit systems, with specific measurement scenario. We will discuss facet inequalities and a set of other inequalities. We will see that same phenomenon happens. A non-facet inequality is violated maximally by a maximally-entangled state, while a facet inequality is not.
• We will also see a relation between the entanglement and nonlocality of a class of states – generalized GHZ states.
Bell Polytopes

- In a typical Bell experiment, there can be two or more spatially separated parties - Alice, Bob, Charlie, ... These parties possess subsystems of a physical system. They can measure observables on their subsystems. They can collect data on joint probability distributions \( p(a, b, c, ... | x, y, z, ...) \). Here \( x, y, z \) are observables measured by Alice, Bob, and Charlie; \( a, b, c \) are measurement outcomes. These probability distributions have to satisfy normalization and no-signalling constraints. This reduces the number of these joint probabilities.

- A convex-hull of these probability distributions defines a Bell polytope. The probability distributions inside the polytope are classical distribution. Alternately, one can characterize this polytope by facets. Each facet divides the probability space in two half, and is characterized by an inequality.

- A large fraction, as we will see in concrete examples, of these facets inequalities are just positivity conditions. Rest of the nontrivial inequalities can correspond to Bell-type inequalities.
Let us now look at the familiar case of CHSH inequality. The scenario is \((2, 2, 2)\). There are two parties, two measurements by each party, and two outcomes for each measurements.

For this case there are 16 joint probability distributions \(P(a, b|x, y)\). Taking into account conservation of probability and no-signalling conditions reduce the number to 8.

So we have to consider polytope in 8 dimensions that have 16 vertices. As we will see, this consideration will give rise to 24 facets. The sixteen out of 24 are positivity conditions. Out of the remaining 8 four gives the upper bound and 4 the lower bound. There is only one independent inequality. Other follows from permutations.
Bell-CHSH Polytope

- The whole scenario is characterized by 16 joint probabilities. They satisfy normalization

\[ \sum_{a,b} p(ab|xy) = 1 \quad \forall \quad x, y = 0, 1. \]

- No signalling conditions are

\[ p(a|x) \equiv \sum_{b} p(ab|xy) = 1 \quad \forall \quad a \text{ and } x, y = 0, 1, \]

\[ p(b|y) \equiv \sum_{a} p(ab|xy) = 1 \quad \forall \quad b \text{ and } x, y = 0, 1. \]

- There are 4 normalization constraint and 12 no-signaling constraint. But these constraints are not all independent. Using normalization constraint we can reduce the no-signaling condition by 4. Therefore, there will be total 8 independent constraints. These 8 constraints will reduce the joint probabilities space to 8.
Bell-CHSH facets

• These probability space can be represented as

\[ p = [p(a_1), p(a_2), p(b_1), p(b_2), p(a_1b_1), p(a_1b_2), p(a_2b_1), p(a_2b_2)], \]

where \( p(a_x) = p(1|x), p(b_y) = p(1|y) \) and \( p(a_xb_y) = p(11|xy) \).

• As \( a_1, a_2, b_1 \) and \( b_2 \) can take two different diachromatic values, there will be total 16 different points in the above said probability space. So, the CHSH-polytope is 8 dimensional and described by 16 vertices. The polytope described in terms of vertices known as V-representation. One can find the faces of the polytope from this description by using a standard algorithm. There will be 24 facets as follows,

\[
\begin{align*}
    p(a_ib_j) & \geq 0, \quad i = 1, 2 \quad \text{and} \quad j = 1, 2 \\
    -p(a_i) + p(a_ib_j) & \leq 0, \quad i = 1, 2 \quad \text{and} \quad j = 1, 2 \\
    -p(b_j) + p(a_ib_j) & \leq 0, \quad i = 1, 2 \quad \text{and} \quad j = 1, 2 \\
    p(a_i) + p(b_j) - p(a_ib_j) & \leq 1, \quad i = 1, 2 \quad \text{and} \quad j = 1, 2
\end{align*}
\]
Bell-CHSH facets

- The nontrivial facets are

\[ p(a_1) + p(b_2) - p(a_1 b_1) - p(a_1 b_2) + p(a_2 b_1) - p(a_2 b_2) \leq 1 \]
\[ p(a_1) + p(b_1) - p(a_1 b_1) - p(a_1 b_2) - p(a_2 b_1) + p(a_2 b_2) \leq 1 \]
\[ p(a_2) + p(b_2) + p(a_1 b_1) - p(a_1 b_2) - p(a_2 b_1) - p(a_2 b_2) \leq 1 \]
\[ p(a_2) + p(b_1) - p(a_1 b_1) + p(a_1 b_2) - p(a_2 b_1) - p(a_2 b_2) \leq 1 \]
\[ -p(a_1) - p(b_2) + p(a_1 b_1) + p(a_1 b_2) - p(a_2 b_1) + p(a_2 b_2) \leq 0 \]
\[ -p(a_1) - p(b_1) + p(a_1 b_1) + p(a_1 b_2) + p(a_2 b_1) - p(a_2 b_2) \leq 0 \]
\[ -p(a_2) - p(b_2) - p(a_1 b_1) + p(a_1 b_2) + p(a_2 b_1) + p(a_2 b_2) \leq 0 \]
\[ -p(a_2) - p(b_1) + p(a_1 b_1) - p(a_1 b_2) + p(a_2 b_1) + p(a_2 b_2) \leq 0. \]

- The last eight inequalities are the famous CH inequalities and are equivalent to CHSH inequalities. However, there is only one independent inequality. Here four give the lower, and four the upper bound. Out of the four, three can be obtained by permutations.
Multipartite States

- In the case of multipartite states comparing the entanglement of even two states is not straightforward.
- The nature of entanglement is not very well understood.
- There is a long history of finding inequalities for multipartite states. We will particularly focus on three-qubit states.
- There are various issues with a number of popular inequalities.
- Mermin inequality is
  \[ I_M = A_1B_1C_2 + A_1B_2C_1 + A_2B_1C_1 - A_2B_2C_2 \leq 2. \]
- Svetlichny Inequality is
  \[ I_S = A_1D_1C_1 + A_1D_2C_2 + A_2D_2C_1 - A_2D_1C_2 \leq 4, \]
  where \( D_1 = B_1 + B_2 \) and \( D_2 = B_1 - B_2 \).
- These inequalities have problems with some states that have genuine tripartite entanglement, in particular some of generalized GHZ (GGHZ) states.
Multipartite Facet Inequalities

- In the case of three parties when each party makes two measurements of diachromatic observables, there are 46 facet inequalities (Sliwa). Mermin inequality is one of these facet inequalities. One of these facet inequality is to be violated a three-qubit pure state.
- There is another set of 185 inequalities. At least one of these will be violated by a genuinely entangled tripartite pure state. Svitlichny inequality is one of these inequalities. Some of these inequalities have been used for various purposes.
- However, in a minimal scenario, two parties, Alice and Bob make two measurements of diachromatic observables, while Charlie makes only one measurement of a diachromatic observable. Then one has only one nontrivial facet inequality.
- Similarly, one can consider the cases, when Alice or Bob make only one measurement. So we have three independent facet inequalities for our scenario.
- In terms of CHSH inequality of those who make two measurements each, this set can be written as,

\[-6 \leq I_{CHSH} + I_{CHSH}A_1 - 2A_1 \leq 2,
-6 \leq I_{CHSH} + I_{CHSH}B_1 - 2B_1 \leq 2,
-6 \leq I_{CHSH} + I_{CHSH}C_1 + 2C_1 \leq 2.\]
Outline

Introduction

Quantum Entanglement

Quantum Nonlocality

Quantum Communication

Quantum Cryptography

Conclusions
Quantum Communication

• There are of course many tasks that one can carry out with the assistance of entanglement. Some of these tasks are only possible with quantum systems.
• We have mentioned Bell inequality violation and quantum cryptography protocols.
• Study of entanglement took off with the introduction of many communication protocols and possibility of more powerful algorithms for some computing tasks.
• In communication protocols, some early ones that you may have seen are teleportation, superdense coding, entanglement swapping, secret sharing etc. In most of these protocols one party, say Alice, can communicate quantum states, classical information, to Bob either directly, or with the help of one or more parties.
• We will discuss three of the earliest communication protocols that underlined the importance of quantum resources. These three protocols are
  - Teleportation
  - Superdense Coding
  - Entanglement Swapping
• Suppose Alice wishes to send an unknown state to Bob. One way would be to send the system in that state to Bob. Can one do better?
• It turns out that if Alice and Bob share an entangled state, then using local operation and classical communication (LOCC), Alice can send an unknown state to Bob.
• In the original protocol, Alice transmits unknown one-qubit state using a Bell state as a resource and using two classical bits of information.
• We now know that any entangled bipartite pure state is suitable for teleportation, which may be probabilistic.
• If we have a multipartite state as a resource, there exits many possibilities for exact teleportation.
• If there are only two parties, Alice can teleport a state of one or more qubits with varying number of terms.
• We will discuss the original protocol.
Suppose Alice wishes to transmit following unknown qubit state to Bob

\[ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \]

The quantum resource is a Bell state

\[ |\varphi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]

Alice has one qubit and Bob has the other qubit of this shared entangled state. Alice has a second qubit in the state \(|\psi\rangle\).

In this protocol, Alice makes a measurement on her two qubits and conveys the outcome to Bob. Bob then makes a suitable unitary transformation on his qubit to convert its state to \(|\psi\rangle\).

To see this, let us look at the state of the combined three qubits.
Teleportation

The state of three qubits can be written as

$$ |\psi\rangle|\varphi^+\rangle = \frac{1}{\sqrt{2}} \left( (\alpha |0\rangle + \beta |1\rangle) (|00\rangle + |11\rangle) \right) $$

$$ = \frac{1}{2} \left[ |\varphi^+\rangle (\alpha |0\rangle + \beta |1\rangle) + |\varphi^-\rangle (\alpha |0\rangle - \beta |1\rangle) + |\psi^+\rangle (\beta |0\rangle + \alpha |1\rangle) + |\psi^-\rangle (\beta |0\rangle - \alpha |1\rangle) \right] $$

If Alice makes a measurement in Bell basis, the state of Bob’s qubit will be $(\alpha |0\rangle \pm \beta |1\rangle)$, or $(\beta |0\rangle \pm \alpha |1\rangle)$.

After making the measurement, the Alice will send two classical bits of information to Bob, encoding her measurement result. On the basis of this communication, Bob will apply suitable unitary transformation to his qubit to recover the state $|\psi\rangle$. 
Teleportation

- Following transformations will do the job for Bob

\[
\sigma_0 (\alpha|0\rangle + \beta|1\rangle) = |\psi\rangle \\
\sigma_3 (\alpha|0\rangle - \beta|1\rangle) = |\psi\rangle \\
\sigma_1 (\alpha|1\rangle + \beta|0\rangle) = |\psi\rangle \\
\sigma_3\sigma_1 (\alpha|1\rangle - \beta|0\rangle) = |\psi\rangle
\]

- **Summary** of the protocol - Using a shared Bell state, Alice can send an unknown quantum state to Bob by making local two-qubit projective measurement and 2 bits of classical communication to Bob. Bob has to make a local unitary transformation.

- What if the resource state is not maximally entangled? In that case perfect teleportation is not possible.

- It has been shown that all entangled pure states are suitable for teleportation. Teleportation fidelity will be more than \(\frac{2}{3}\). It is a bit more complicated, but even state with hidden locality can be used for teleportation.
Superdense Coding

- This protocol shows how one can use entanglement to send more classical information. (Bennet and Wisner)
- Alice has a qubit and she wishes to transmit as much classical information as possible by sending her qubit to Bob.
- If her qubit is not entangled with Bob’s qubit, she can send at most one classical bit of information.
- Since Alice has only a two-state system, she can encode at most one classical bit in its state. For example, she can encode classical bit 0 in the state $|0\rangle$, and 1 in the state $|1\rangle$. She then sends her qubit to Bob.
- Bob makes a measurement in $\{|0\rangle, |1\rangle\}$ basis on Alice’s qubit. If he obtains $|0\rangle$, he gets the message encoded in 0, or if he obtains $|1\rangle$, he gets the message encoded in 1.
- On the other hand, if Alice shares a Bell state with Bob, she can send two classical bits to Bob by sending her qubit to him.
Superdense Coding

- Here is what to do
  
  $00 : \sigma_0 \otimes \sigma_0 |\varphi^+\rangle = |\varphi^+\rangle$
  
  $01 : \sigma_1 \otimes \sigma_0 |\varphi^+\rangle = |\psi^+\rangle$
  
  $10 : \sigma_3 \sigma_1 \otimes \sigma_0 |\varphi^+\rangle = |\psi^-\rangle$
  
  $11 : \sigma_3 \otimes \sigma_0 |\varphi^+\rangle = |\varphi^-\rangle$

- Alice makes one of the four local transformations on her qubit to encode classical information. As we see above she can encode now two classical bits.

- After the transformation, she sends her qubit to Bob.

- Bob then makes a Bell measurement on both the qubits and decodes the Alice’s message.

- Since, using entanglement, Alice has been able to encode more classical information, this protocol is called superdense coding.

- All entangled pure states can be used for superdense coding.
Entanglement Swapping

- In this protocol, there are four parties Alice, Bob, Charlie and Denis with one qubit each. Initially, Alice and Bob share a Bell state; Charlie and Denis share a Bell state. After the execution of the protocol, Alice and Charlie will share a Bell state; so will Bob and Denis. So entanglement is swapped.

- In the beginning the combined state of four qubits is:

\[ |\chi\rangle_{ABCD} = |\varphi^+\rangle_{AB} |\varphi^+\rangle_{CD} \]

- A and B are entangled, but A and C are not.

- Now let A and C make a joint Bell measurement on their qubits. This will lead to A and C getting entangled and A and B disentangled.

- So just a measurement will create entanglement between two qubits, by destroying entanglement elsewhere.
Entanglement Swapping

- To see this let us rewrite the above state

$$|\chi\rangle_{ABCD} = |\varphi^+\rangle_{AB}|\varphi^+\rangle_{CD}$$

$$= \frac{1}{2} [ |\varphi^+\rangle_{AC}|\varphi^+\rangle_{BD} + |\varphi^-\rangle_{AC}|\varphi^-\rangle_{BD} +$$

$$|\psi^+\rangle_{AC}|\psi^+\rangle_{BD} + |\psi^-\rangle_{AC}|\psi^-\rangle_{BD} ]$$

- If Alice makes a measurement in Bell basis, \{\{\varphi^\pm, |\psi^\pm\rangle\}\}, then after the measurement, the states of AC and BD would be entangled. So entanglement has been swapped from “AB and CD” to “AC and BD”.
Multi-qubit states

- In the standard teleportation protocol, Alice wishes to transmit an unknown quantum state to Bob using an entangled state as a resource and classical communication.

- In the original protocol, Alice transmits unknown one-qubit state using a Bell state as a resource and using two classical bits of information.

- We now know that any entangled bipartite pure state is suitable for teleportation, which may be probabilistic.

- If we have a multipartite state as a resource, there exit many possibilities for exact teleportation.

- If there are only two parties, Alice can teleport a state of one or more qubits with varying number of terms. We may not be interested in maximal teleportation.

- So for example, with a six qubit resource state, we may not be necessarily interested in teleporting an arbitrary three qubit state with eight terms, but fewer terms – between 2 and 8.
Maximal Teleportation and Superdense Coding

- For a given number of qubits, there exist states that can be used to carry out *maximal* teleportation.
- For maximal teleportation, an $n$-qubit state would allow us to teleport an unknown arbitrary $\frac{n}{2}$-qubit state, when $n$ is even and $\frac{(n-1)}{2}$-qubit state, when $n$ is odd.
- Such a state will be maximally entangled from the point of view of teleportation.
- For maximal superdense coding, a $n$-qubit state would allow us to transmit $n$ classical bits of information by sending $\frac{n}{2}$ qubits when $n$ is even and $\frac{(n+1)}{2}$ qubits when $n$ is odd.
- A $n$-qubit state that allows maximal superdense coding would be a TMES with respect to superdense coding.
- There are prescriptions to construct TMES for teleportation and superdense coding.
Maximal Teleportation and Superdense Coding

- Easiest way to teleport an arbitrary unknown $d$-qubit state is to use $d$ Bell states. Then each qubit of the $d$-qubit state can be teleported using one of the Bell states. This means that for the larger number of qubits, one could take a direct product of Bell states as the quantum resource. However, as these resource states are not genuinely multipartite entangled states, this resource is not very interesting.

- However, by a multinary unitary transformation, one could convert these unentangled states into entangled states. Since the teleportation protocol is not affected by these unitary transformations, these states with genuine multipartite entanglement also serve the purpose of teleporting a $d$-qubit state.

- One can see this as following. Suppose we wish to teleport an arbitrary unknown $n$-qubit state $|\psi\rangle$ using the $m$-qubit state $|R\rangle$ as a quantum resource ($m \geq 2n$). If the teleportation is successful, then we would be able to write,

$$|\psi\rangle_{a_1a_2\ldots a_n}|R\rangle_{b_1b_2\ldots b_m} = \frac{1}{2^n} \sum_{i=1}^{2^{2n}} |O^i\rangle_{a_1a_2\ldots a_n b_1b_2\ldots b_{m-n}} V^{i\dagger}_{b_{m-n+1}b_{m-n+2}\ldots b_m} |\psi\rangle_{b_{m-n+1}b_{m-n+2}\ldots b_m}.$$
Maximal Teleportation and Superdense Coding

- Here subscripts are particle labels. $|O^i\rangle$ are a set of orthogonal states and $V^i$ are unitary operators. Alice would need to send $2n$ cbits of information to Bob to teleport a $n$-qubit state.

- If we apply an unitary transformation $I_{a_1a_2\ldots a_n} \otimes U_{b_1\ldots b_{m-n}}$ to the both side of the equation, then this transformation would convert the set of orthogonal states $|O^i\rangle$ to another such a set. So the teleportation would still be possible, but with a different quantum resource state, $U_{b_1\ldots b_{m-n}}|R\rangle_{b_1b_2\ldots b_m}$.

- Therefore, if we know a quantum resource state (e.g., a product of Bell states) that would allow the teleportation of an arbitrary unknown $n$-qubit state, then we can find another resource state by applying appropriate unitary transformation to it. Alice can apply this unitary transformation. (Instead of Alice, Bob can also do it.)

- Interestingly, these entangled states can also be used for maximal superdense coding for even number of qubits. This may not be surprising because of the close relationship between the two tasks.
Two-qubit TMESs

- one can construct TMESs for any number of qubits. For illustration, we will discuss these states for two, three, and four qubits.
- For a two-qubit system, the TMESs are well known. These are maximally entangled states, Bell states,
  \[
  |\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)
  \]
  \[
  |\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).
  \]
- Using these states, one can carry out the conventional teleportation and superdense coding. One can teleport one-qubit state perfectly.
- One can also transmit two cbits by sending one qubit. Therefore, with Alice and Bob one qubit each, both protocols can be carried out maximally.
- Alice can convert these states into one another by applying unitary transformations \(\{\sigma_0, \sigma_1, i\sigma_2, \sigma_3\}\) on her qubit. Here \(\sigma_0\) is a \(2 \times 2\) unit matrix, and \(\sigma_1, \sigma_2, \sigma_3\) are Pauli matrices.
- As a point of interest, these Bell states can be obtained from the product states \{\(|+\rangle|0\rangle, |−\rangle|0\rangle, |+\rangle|1\rangle, |−\rangle|1\rangle\} by applying the CNOT unitary operator \(U^{CN}\). As this operator acts on two qubits, it can entangle them.
Three-qubit TMESs

• None of the three-qubit states can be used to teleport an unknown arbitrary two-qubit state. Furthermore since the Hilbert space of a three-qubit system is only eight-dimensional, one cannot transmit four cbits by transmitting two qubits. At most three cbits could be transmitted.

• To obtain an TMES, one can start with the tensor product of a Bell state and a computational basis state. In the case of three-qubit system, e.g., we could have,

\[ U_{13}^{\text{CN}} |\varphi^+\rangle_{12} |0\rangle_3 = \frac{1}{\sqrt{2}} U_{13}^{\text{CN}} (|000\rangle_{123} + |110\rangle_{123}) \]

\[ = \frac{1}{\sqrt{2}} (|000\rangle_{123} + |111\rangle_{123}). \]

This is a GHZ state. For a three-qubit systems it is a TMES for the tasks of superdense coding and teleportation.

• The CNOT operator is

\[ U^{\text{CN}} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \]

• By applying other suitable unitary operator, one can construct other three-qubit TMESs.
Maximal Teleportation and Superdense Coding

- Let us now come to a non-trivial situation of four-qubits.
- At different times, arguments have been put forward for different states to be maximally entangled. Some of these states are:

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)
\]

\[
|\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle)
\]

\[
|\chi\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)
\]

\[
|HS\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle + \omega(|1010\rangle + |0101\rangle) + \omega^2(|1001\rangle + |0110\rangle))
\]

Here \( \omega = e^{\frac{2\pi i}{3}} \).

- The GHZ-state has all one qubits in completely mixed state. The \(|\Omega\rangle\) state show certain features of entanglement which are persistent. The \(|\chi\rangle\) state can be used for what we call maximal teleportation and superdense coding. The \(|HS\rangle\) state has maximized two-qubit correlations.
Maximal Teleportation and Superdense Coding

- But of these states only $|\Omega\rangle$ and $|\chi\rangle$ are TMES wrt to teleportation and superdense coding.
- We can construct these states by applying suitable unitary operators to a product of Bell states. For example,

$$U^{CN}_{13} |\varphi^+\rangle_{12} |\varphi^+\rangle_{34} = \frac{1}{2} U^{CN}_{13} (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{1234}$$

$$= \frac{1}{2} (|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle)_{1234}.$$

- One can use a set of sixteen linearly independent set of unitary operators to construct (up to locally unitarily equivalent) set of TMESs.
- This procedure can be extended to any number of qubits. For an even number of qubits, start with appropriate number of Bell states. For odd number of qubits, one would also need an additional computational basis state also.
Less is More

- In the case of multipartite system, in some situations one can do more communication with less entangled state.
- We consider teleportation, superdense coding, Quantum Key distribution to illustrate this feature.
- In the case of teleportation, one needs special amount of entanglement for the teleportation of the state of a $n$-qubit system with $m$ terms.
- Since the state requires specific values of entanglement, it is not surprising that some time more entangled state is not useful.
- We also exhibit some states that can be used for superdense coding but are not useful for teleportation.
- Although we would not discuss in detail, but the phenomenon of “Less is More” also appears for many protocols for multipartite states, e.g., remote state preparation, cooperative teleportation, cooperative superdense coding, etc.
Less is More

- Our discussion will be primarily for four-qubit states, but conclusions will be valid for $n$-qubit states.
- For teleportation and superdense coding, there are only two parties, Alice and Bob.
- With a two-qubit resource state, one can transmit at most one-qubit state, or two classical bits.
- With a resource state with more qubits, one can teleport multi-qubit unknown states and send more than two classical bits.
- For example, if we have a suitable four-qubit resource state, then one can teleport not only one-qubit state, but also a two-qubit state with two, three, or four superposed terms.
- One can also densecode three or four classical bits.
- One can carry out cooperative teleportation and cooperative superdense coding and many other protocols.
We consider the following four-qubit entangled states

\[ |\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \]
\[ |\text{W}\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \]
\[ |\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle) \]
\[ |Q_4\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) \]
\[ |Q_5\rangle = \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \]
von Neumann Entropy

To see how entangled these states are, we compute entropy of all subsystems. In a bipartite partition, both subsystems will have identical entropies. Therefore, if we split the system in particle 1 on one hand and particles 2, 3 and 4 on the other, then the entropy of the particle 1, $S(\rho_1)$ will be the same as that of particle 234 subsystem, $S(\rho_{234})$.

<table>
<thead>
<tr>
<th></th>
<th>$S(\rho_1)$</th>
<th>$S(\rho_2)$</th>
<th>$S(\rho_3)$</th>
<th>$S(\rho_4)$</th>
<th>$S(\rho_{12})$</th>
<th>$S(\rho_{13})$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<tr>
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<td>0.81</td>
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<tr>
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<td>0.81</td>
<td>0.81</td>
<td>1.5</td>
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<tr>
<td>$Q_5$</td>
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<td>1</td>
<td>1</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

_Table 1: Entropies of the subsystems_
Teleportation

- In such a case we may not need a TMES for the teleportation, a “less” entangled state may be enough.
- Let us now consider the four-qubit states to illustrate the various possibilities. We will also see that a state with more entanglement is sometimes not suitable for the teleportation.
- Let us now consider the teleportation of the state of the most general two-qubit state.

\[ |\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle. \]

- One can check that, one can construct a suitable measurement basis only in the case of \( \Omega \)-state as a resource with Alice holding the qubits ‘12’ or ‘13’ only. These partitions have entropy as 2. Alice makes a four-particle measurement in a suitable basis and sends four classical bits of information to Bob. Bob then can convert his two-qubit state to that of unknown state with suitable unitary transformations.
Teleportation

• Let us know consider a state with only two terms

\[ |\psi\rangle = \alpha |00\rangle + \beta |11\rangle. \]

• One can check that GHZ-state and Ω-state can be used for teleporting this state to Bob with two classical bits of communication.

• However, it is not possible to construct suitable measurement basis, to teleport this state using Q4-state or Q5-state.

• If we look at the entropy table for 2:2 partitions – ‘12’, ‘13’ and ‘14’, we see that the Q4 and Q5 states are more entangled, but they are not suitable for teleportation.

• So, more entanglement does not necessarily mean more communication.
• Let us now discuss the teleportation of a one-qubit state using the considered four qubit states.
• All partitions of the GHZ-state and Ω-state are suitable for teleportation.
• None of the partitions of the W-state are suitable.
• Only those partitions of the Q4 and Q5 states are suitable for teleportation where Bob’s qubit has entropy as one.
• Basically, if the Bob’s qubit has entropy one, the state with particular partition is suitable for teleportation, otherwise it is not.
Condition for Teleportation

- Question we pose is: \textbf{if we are given a }n\text{-qubit state with }m\text{ terms to teleport, what kind of resource state is needed for exact teleportation, i.e., teleportation with unit probability and unit fidelity?} \textbf{?}

- Or, in reverse, \textbf{given a resource state, what are the states that can be teleported using this resource?}

- The answer to this question helps us to see why sometimes a more entangled state is less suitable.

- Answer to this question is - \textbf{if we wish to teleport a }n\text{-qubit state with }m\text{ terms, then we should be able to distribute resource states qubits in such a way such that Bob’s }n\text{ qubits have entropy }\log_2 m\text{.}

- \textbf{Given a resource state, we can compute entropy of all the partitions. If there is a partition where Bob’s }n\text{ qubits have entropy as }\log_2 m\text{, then the state with }m\text{ terms can be teleported with that partition.}
Superdense Coding

- We are considering four-qubit states. So Alice can send at most four classical bits to Bob. She has option of sending one, two, or three qubits.

- With GHZ state, by sending $n$ qubits, Alice can transmit $n + 1$ classical bits. So deterministic dense coding is always possible, but the maximal dense coding is not possible. Therefore, GHZ state is not a TMES with respect to superdense coding.

- The cluster state is the best. Two of its partitions ‘12’ and ‘13’ allow us to transmit four classical bits by sending two qubits. This is because these partitions give rise to entropy of two for the subsystems. Other partitions behave like GHZ state. So, by sending $n$ qubits, Alice can transmit $n + 1$ classical bits.

- These states are useful, since the subsystems have nice values of entropies.
Superdense Coding

- **The Q4-state exhibit the phenomena of less communication with more entanglement**

- If we consider the partitions ‘13’ and ‘14’, the subsystems have entropy 1.22, which is more than GHZ and W-states. But if we apply local unitaries on these two qubits, we get four orthogonal states. So if we use these partitions, one can transmit only two classical bits by sending two qubits. So there is no superdense coding in these situations for Q4-state.

- However, partition ‘12’, with entropy 1.5 is useful for dense coding. As we saw, this partition was not useful for teleportation.

- In the case of sending one or three qubits to Bob, the protocol will work when Alice either sends qubits ‘134’ or qubit ‘2’. In all other cases, the protocol would not work, since the subsystems have entropy less than one.
Cooperative Teleportation and Superdense Coding

- In these protocols, there are more than two parties. Qubits are distributed among these parties.
- Alice teleports a state to Bob, or dense codes the classical information for Bob. However, Bob can receive the state or the classical information only with the cooperation of the other parties.
- In the case of these protocols also, GHZ-state and Ω-state are more successful than the Q4 and Q5 states.
- One can construct states with higher number of qubits, say six qubits. One can again see the phenomena of ‘less is more’, by distributing more than one qubit to each party.
Two-qubit mixed states

- Case of mixed states is more involved, due to the existence of hidden nonlocality and existence of many parameters (15 parameters which can be reduced to 9).
- Due to the existence of many parameters, one may need to consider many functions of these parameters to fully characterize its nonlocality and optimize a protocol.
- For a general mixed state, analysis can be complicated. To reduce this complicacy, we will consider a simpler class of states to illustrate some of the issues.
- The class that we will consider is called X-states. Some of the well known states fall into this class.
Two-qubit mixed states

• The parametric form of an arbitrary two-qubit X-state of a bipartite system can be represented as follow:

\[
\rho_{AB} = \begin{pmatrix}
\cos^2 \theta & 0 & 0 & \sqrt{x}e^{i\mu} \\
0 & \sin^2 \theta \cos^2 \phi & \sqrt{y}e^{i\nu} & 0 \\
0 & \sqrt{y}e^{-i\nu} & \sin^2 \theta \sin^2 \phi \cos^2 \psi & 0 \\
\sqrt{x}e^{-i\mu} & 0 & 0 & \sin^2 \theta \sin^2 \phi \sin^2 \psi
\end{pmatrix}
\]  

(3)

with \(\theta, \phi, \psi \in [0, \frac{\pi}{2}]\), \(x, y \geq 0\) and \(\mu, \nu \in [0, 2\pi]\). However these conditions are not enough to make Eq. (3) a valid density matrix. Further constraints \(x \in [0, \mathcal{H}]\) and \(y \in [0, \mathcal{G}]\) are required to make it positive semidefinite. Here we define \(\mathcal{H} = \sin^2 \theta \cos^2 \theta \sin^2 \phi \sin^2 \psi\) and \(\mathcal{G} = \sin^4 \theta \cos^2 \phi \sin^2 \phi \cos^2 \psi\).

• A two-qubit X-state will be of rank one if \((x = \mathcal{H}, y = 0, A = 0)\) or \((x = 0, y = \mathcal{G}, B = 0)\), where \(A = \sin^2 \theta (1 - \sin^2 \phi \sin^2 \psi)\) and \(B = 1 - A\). A rank two X-state can be parametrized as \((x < \mathcal{H}, y = 0, A = 0)\) or \((x = 0, y < \mathcal{G}, B = 0)\) or \((x = \mathcal{H}, y = \mathcal{G}, AB > 0)\). Conditions \((x < \mathcal{H}, y = \mathcal{G}, A > 0)\) or \((x = \mathcal{H}, y < \mathcal{G}, B > 0)\) parametrize a two-qubit X-state of rank three. A rank four X-state can be parametrized as \((x < \mathcal{H}, y < \mathcal{G}, AB > 0)\).
Two-qubit mixed states

- the following expressions for purity and concurrence for an arbitrary X-state have been obtained,

\[ \mathcal{P} = 1 - 2(AB + G - y + H - x) \]  \hspace{1cm} (4)
\[ C = 2 \text{ max } [\sqrt{x} - \sqrt{G}, \sqrt{y} - \sqrt{H}] \]  \hspace{1cm} (5)

- Optimal fidelity of teleportation is

\[ F = \frac{1}{2} \left[ 1 + \frac{1}{3} \text{Tr} \sqrt{T^\dagger T} \right]. \]  \hspace{1cm} (6)

where the elements of the matrix \( T \) are defined as \( t_{mn} = \text{Tr}[\rho(\sigma_n \otimes \sigma_m)] \) and \( m, n = (1, 2, 3) \).

- For the general X-state, this is,

\[ F = \frac{1}{6} \left[ 3 + 2\sqrt{(\sqrt{x} + \sqrt{y})^2 + 2\sqrt{(\sqrt{x} - \sqrt{y})^2 + \sqrt{\left(\cos^2 \theta - \sin^2 \theta (\cos^2 \phi + \sin^2 \phi \cos 2\psi)\right)^2}} \right]. \]  \hspace{1cm} (7)
Two-qubit mixed states

- In the case of rank-one and some of the rank-two X-states, the relationship between concurrence, purity, and fidelity is quite simple:

\[ F = \frac{1}{3} (2 + C). \]

- We can see that for any pure entangled state fidelity is larger than \( \frac{2}{3} \). But it is not the case always for a mixed state.

- As we consider higher rank X-states, there is something unexpected. A state with larger purity and concurrence, may have comparatively smaller fidelity.

- For such states, fidelity changes monotonically with respect to functions of parameters - other than concurrence and purity. A state with smaller concurrence and purity, but larger value of one of these functions has larger fidelity.

- These functions, characterize nonlocal classical and/or quantum properties of the state that are not captured by purity and concurrence alone.

- The concurrence is not enough to characterize the entanglement properties of a two-qubit mixed state.
Two-qubit mixed states

- Third rank X-state of first kind is characterized by, $x < H, y = G, A > 0$.
- For this state, the fidelity is

$$\mathcal{F} = \frac{1}{6} \left[ 4 + 2C - (1 + e - 4f) V_1(P, C, d, f) \right],$$

where,

$$d = 1 - \sin^2 \phi \sin^2 \psi + \sin^4 \phi \sin^4 \psi,$$

$$e = \cos^2 \phi + \sin^2 \phi \cos 2\psi,$$

$$f = \cos \phi \sin \phi \cos \psi,$$

$$V_1(P, C, d, f) = \frac{(1 - fC) \pm \sqrt{(1 - fC)^2 - (1 - P + C^2/2)(2d + 2f^2)}}{(2d + 2f^2)}.$$
Figure: Variation of optimal fidelity for $d = \frac{3}{4}$, $e = 0$ and $f = 0$ (i) $C = 0.2$ and (ii) $P = 0.7$. 
Variation of optimal fidelity

Figure: Variation of optimal fidelity for $\mathcal{P} = .7$ and $\mathcal{C} = 0.2$ (i) $f = 0$ and (ii) $e = 0$. 
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You may have heard about the famous BB84 quantum cryptography protocol (Bennett-Brassard, 1984). The protocol involves superposed states and transmitting quantum systems. This protocol did not require entanglement. One does measurement in computational or Hadamard basis.

It was Ekert in 1991, who used entangled states for Alice and Bob to establish a secret key. In the original version, Alice and Bob shared many copies of a Bell state.

So it is the ability of a quantum system to be in a superposed state that is making quantum systems more powerful resources.

We will first discuss a basic cryptography protocol using classical resources. Then we will discuss BB84 protocol. This protocol uses superposition.
Cryptography

• One of the most important application of quantum information rules is in the area of quantum cryptography.

• **Goal of cryptography is to communicate information by Alice to Bob without being compromised by Eve.** The content of information is to remain secret.

• The process is as follows:
  
  • Encrypt the message by using a key $K$, i.e.

    $$ E_K(M) = C $$

    Using a key $K$, one can encrypt a message $M$, and gets the cipher $C$. The key, $K$, is known to only Alice and Bob.

  • $C$ is sent to Bob.

  • Bob decrypts the message using the key $K$, i.e.

    $$ D_K(C) = M $$
One-Time Pad

- let us illustrate this scheme by an example. Alice will encrypt the message as:
  
  The message  \( M : 0 \ 0 \ 1 \ 0 \ 1 \)  
  The Key  \( K : 1 \ 1 \ 1 \ 0 \ 0 \)  
  \[ C : 1 \ 1 \ 0 \ 0 \ 1 \]

- Here \( E_K \) is \( M \oplus K \mod 2 \).

- Bob can decrypt using the same key
  
  \[ 1 \ 1 \ 0 \ 0 \ 1 \oplus 1 \ 1 \ 1 \ 0 \ 0 = 0 \ 0 \ 1 \ 0 \ 1 \]
One-Time Pad

- If Alice and Bob have met in the past and have agreed to use a specific key, they can safely send the message as the key is only known to them. This is an example of Private Key Distribution.

- However, if Alice and Bob have never met, so a key has to be communicated over a public channel, then in principle, the key can be intercepted by a spy, Eve, without Alice and Bob knowing about it.

- Use of QM resources, allows for safe Quantum Key Distribution (QKD) using a public channel.

- This is possible because any measurement made on a quantum system can change it (unlike a classical state). This gives a means to Alice and Bob to realize that their secret is not safe.

- The first popular QKD protocol was devised by Bennett and Bassard in 1984. This protocol is known as BB84 protocol.
BB84 Protocol

• In this protocol, Alice uses one of four non-orthogonal states to communicate 0 or 1 to Bob. Advantage of using non-orthogonal states is that Eve cannot discriminate them perfectly.

• This is how the protocol runs:

• Alice prepares her qubits in any of four states \{\ket{0}, \ket{1}\} and \{\ket{+}, \ket{-}\}. These two bases allow Alice to encode \{0, 1\} in two different ways

  0 : \ket{0}, \ket{+}

  1 : \ket{1}, \ket{-}

• Alice wishes to send a string of bits \(x_1 \ldots \ldots x_n\) to Bob. For example

  Bit : \(x_1 \ x_2 \ x_3 \ x_4 \ x_5\)
  Value : 0 1 1 0 1
**BB84 Protocol**

- In order to prevent Eve from reading the bits, Alice chooses to encode each bit in either computational basis, \{\ket{0}, \ket{1}\}, or Hadamard basis, \{\ket{+}, \ket{-}\}. For example

  \[
  \begin{array}{c|c|c|c|c|c|}
  \text{Value} & 0 & 1 & 1 & 0 & 1 \\
  \text{Quantum Encoding} & \ket{0} & \ket{-} & \ket{1} & \ket{+} & \ket{-} \\
  \end{array}
  \]

- Alice now sends the encoded qubits to Bob. On receiving the qubits, Bob measures each qubit using either computational basis or Hadamard basis. For example

  \[
  \begin{array}{c|c|c|c|c|c|}
  \text{Value} & 0 & 1 & 1 & 0 & 1 \\
  \text{Alice’s basis} & C & H & C & H & H \\
  \text{Bob’s basis} & H & C & C & H & H \\
  \text{In agreement} & \text{No} & \text{No} & \text{Yes} & \text{Yes} & \text{Yes} \\
  \end{array}
  \]
BB84 Protocol

- Both of them publicly declare their sequence of measurement basis, but not the results. In this way, they can find out, when they have used the same basis to encode and decode. They keep the bit, where they have used the same basis, and discard when their basis differ.

- Remaining sequence of bits (here 101) can be used as a key. The key would be known to Alice and Bob only.

- If Eve happens to intercept the encoded qubit where preparation and measurement basis agree, with sufficiently large number of qubits, Eve’s presence can be detected.
Eckert Protocol

- In the Eckert protocol, instead of qubits being transmitted, Alice and Bob share maximally entangled states - Bell states.
- Alice and Bob would each choose a random bases to measure on their qubits. For example, they could use computational and Hadamard bases.
- For each measurement where Alice and Bob use the same bases, they should expect opposite results in the case of singlet state.
- If Alice and Bob both interpret their measurements as bits as before, they each have a bit string which is the binary complement of the other. Either party could invert their key and they would thus share a secret key.
- The presence of an eavesdropper can be detected by examining the qubits for which Alice and Bob chose different bases for measurement.
- Alice and Bob can use the measurement statistics of these cases to check the violation of Bell-CHSH inequality. If inequality is not violated, an eavesdropper may be present.
Quantum Key Distribution

• Let us consider two four-qubit states such as $|GHZ\rangle$ and $|Q_4\rangle$. Let us suppose that the two distant partners Alice and Bob possesses two qubits each.

• From the Table 1, we see that Q4-state is more entangled in these partitions, then the GHZ state.

• Still we will see that in a QKD scheme, Q4-state cannot be used, while GHZ-state can be used.

• To see the protocol, let us rewrite the GHZ-state as

$$|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|00\rangle_{13} \otimes |00\rangle_{24} + |11\rangle_{13} \otimes |11\rangle_{24})$$

The qubits 1 and 3 are held by Alice and the remaining two qubits 2 and 4 are with Bob.
Quantum Key Distribution

- In the Bell basis \(\{|\phi^+, \phi^-, \psi^+, \psi^-\}\}\), \(|\text{GHZ}\rangle_{1234}\) state can be re-written as

\[
\frac{1}{\sqrt{2}} (|\phi^+\rangle_{13} \otimes |\phi^+\rangle_{24} + |\phi^-\rangle_{13} \otimes |\phi^-\rangle_{24})
\]

- In the next step, Alice randomly performs measurements on the particles in either the computational basis or in the Bell basis. The information is encoded by using the two binary digits 0 and 1. In the computational basis (Bell basis), if the measurement outcome is \(|0\rangle_1 \otimes |0\rangle_3\) \((|\Phi^+\rangle_{13})\) then 0 is encoded and if the measurement outcome is \(|1\rangle_1 \otimes |1\rangle_3\) \((|\Phi^-\rangle_{13})\) then 1 is encoded. After Alice’s measurement, Bob also randomly chooses either the computational basis or the Bell basis and then performs measurement on the particles in that basis.

- After this, Alice publicly announces the basis in which she measured the state of the particles but not declare the measurement outcome through a channel.
Quantum Key Distribution

- Bob also announces his measurement basis sequence. The data is kept when their measurement basis matches, otherwise thrown out. In this way, Alice and Bob can establish a quantum key for secret communication. So we see that the $|GHZ\rangle$ state can be used in generating a quantum key.

- To see the usefulness of the Q4-state we rewrite the four qubit state $|Q_4\rangle$ in the computational basis as well as in the Bell basis.

$$|Q_4\rangle = \frac{1}{2} (|00\rangle_{13}|00\rangle_{24} + |00\rangle_{13}|11\rangle_{24} + |10\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|10\rangle_{24})$$

$$|Q_4\rangle = \frac{1}{4} [(2|\phi^+\rangle_{13} + 2|\phi^-\rangle_{13} + |\psi^+\rangle_{13} + |\psi^-\rangle_{13})|\phi^+\rangle_{24} + (|\psi^+\rangle_{13} + |\psi^-\rangle_{13})|\phi^-\rangle_{24} + (|\psi^+\rangle_{13} - |\phi^-\rangle_{13})|\psi^+\rangle_{24} - (|\psi^+\rangle_{13} - |\psi^-\rangle_{13})|\psi^-\rangle_{24}]$$

- It is clear that one cannot generate key even if their basis matches. Therefore, we see that although the four qubit state $|Q_4\rangle$ has more entanglement than $|GHZ\rangle$ state but the former cannot be used in our QKD protocol while the latter can be.
Cooperative QKD

- Cooperative QKD is a protocol in the case of multipartite states.
- Here Alice, Bob, and other parties share an entangled state. Other parties can control whether Alice and Bob can establish a key or not.
- For perfect key distribution, after the measurements by other parties, Alice and Bob must share a Bell state.
- This restricts what kind of states may be suitable for cooperative QKD.
- One requirement is that at least two of the qubits should be completely mixed.
- For three qubits the most general structure of the state is:

$$|\psi\rangle = \sqrt{1/2}[|0\rangle|\phi^+\rangle + |1\rangle(1 \otimes U)|\phi^-\rangle]$$
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Conclusions

- We have discussed some elementary aspects of quantum entanglement.
- We discussed nonlocality and communication protocols using bipartite and multipartite states.
- We also discussed two-qubit mixed states as resource.
- This is a vast area. We discussed only elementary aspects of some areas.
- Ocean is out there for you.