Entanglement and Global Unitary Operations

Effect of change of basis on quantum resources

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Quantum Entanglement

Quantum Entanglement

Let H denote a Hilbert space. Suppose a pure state $|\psi\rangle \in H$. If one can find $|\psi\rangle_A \in H_A$ and $|\psi\rangle_B \in H_B$ (H_A and H_B refer to the subsystems), such that $|\psi\rangle$ can be written as,

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \tag{1}$$

then $|\psi\rangle$ is known to be separable. However, states which cannot be written as in (1) are known as entangled states. A bipartite mixed composite system ρ is said to be in an entangled state if ρ cannot be expressed in a convex combination as

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1 \tag{2}$$

where ρ_i^A and ρ_i^B pertain to the respective subsystems.

Quantum Entanglement

- Consider the state $|\,\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ which belongs to a two qubit system. This state is entangled as there exist no $|\,\psi\rangle_A$ and $|\,\psi\rangle_B$, such that $|\,\psi\rangle = |\,\psi\rangle_A \otimes |\,\psi\rangle_B$.
- In a $3 \otimes 3$ system one can consider the vectors
  $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle)$
  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle$
  $|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle)$
  $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle$
  $|\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)$
  The mixed state $\rho = \frac{1}{4}(I - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|)$ is entangled as it cannot be expressed in convex combination of separable density matrices.

Absolutely Separable States

Absolutely Separable States

1. In order to define separability , a choice of basis is made.
2. In a different basis the state may not be separable any more.
3. Change of basis is brought about by unitary operations.
4. However, local unitaries $U_{AB} = U_A \otimes U_B$ do not create entanglement.
5. Global unitaries create entanglement.
6. There are certain which preserve separability under any global unitary operation. They are termed as Absolutely separable states.

Separability from Spectrum Problem

- Open Problem (E.Knill 2003) "*For a mixed state ρ on an N M dimensional Hilbert space: Are there any factorizations into an N tensor M dimensional space with respect to which the state is not separable ? This depends only on the spectrum of ρ and the problem is to characterize the spectra for which the answer is "NO"* .
- Absolute Separability: They are states that remain separable under all global unitary operations, i.e., $\mathbf{AS} = \{ \sigma_{as} : U\sigma_{as}U^{\dagger}$ is separable $\forall\ U\}$.
- The problem of separability from spectrum was first addressed in the case of $2 \otimes 2$ systems (Verstraete et.al. 2001), where it was shown that σ_{as} is absolutely separable if and only if (iff) its eigenvalues (in descending order) satisfy $\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}$.The result was further extended to $2 \otimes n$ dimensions (Johnston 2013).

Separability from Spectrum Problem

- A closely related problem is the characterization of the states which have positive partial transpose (PPT) from spectrum.
- It was shown (Hildebrand 2007) that $\sigma_{ppt} \in D(H_2 \otimes H_n)$ ($D(X)$ represents the bounded linear operators acting on X) is PPT from spectrum iff its eigenvalues obey $\lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n-2}\lambda_{2n}}$.
- It was shown that for $2 \otimes d$ systems, absolute PPT and absolute separability are equivalent(Johnston 2013).

Detection of states from which entanglement can be generated

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Notations

- Consider density matrices in any arbitrary dimensional bipartite system, i.e., $\rho \in D(H_m \otimes H_n)$.
- $\mathbf{S} = \{\rho : \rho \text{ is separable}\}$ is the set of all separable states.
- $\mathbf{AS} = \{\sigma \in \mathbf{S} : U\sigma U^\dagger \text{ is separable } \forall \text{ unitary operators } U\}$ is the set of all absolutely separable states.
- $\mathbf{AS}$ forms a non-empty subset of $\mathbf{S}$, as $\frac{1}{mn}(I \otimes I) \in \mathbf{AS}$.

Witness to detect non-absolutely separable states

Theorem
AS *is a convex and compact subset of* **S**

The above theorem enables us to define an operator T such that:

$$Tr(T\sigma) \geq 0, \forall \sigma \in \textbf{AS} \tag{3}$$

$$\exists \chi \in \textbf{S} - \textbf{AS}, \quad \text{s.t.} \quad Tr(T\chi) < 0 \tag{4}$$

Therefore, T identifies those separable states χ that become entangled under some global unitary operation.

Examples

- The state

$$\chi_{2\otimes 2} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

is not absolutely separable as detected by the operator

$$T_1 = U_1^{\dagger} W_1 U_1 \tag{6}$$

where

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, W_1 = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

with $c = \frac{1}{\sqrt{2}+1}$

Examples

A general two qudit state: Take the unitary operator

$$U_{d\otimes d} = I - (\frac{\sqrt{2}-1}{\sqrt{2}})A + \frac{1}{\sqrt{2}}B \tag{8}$$

where $A = |00\rangle\langle 00| + |d-1, d-1\rangle\langle d-1, d-1|$ and $B = |00\rangle\langle d-1, d-1| - |d-1, d-1\rangle\langle 00|$, and the mixed separable state

$$\chi_{d\otimes d} = \frac{1}{4}|00\rangle\langle 00| + \frac{3}{4}|d-1, d-1\rangle\langle d-1, d-1| \tag{9}$$

The state $U_{d\otimes d}\chi_{d\otimes d}U^{\dagger}_{d\otimes d}$ is entangled as detected by the witness $W_{d\otimes d} = \frac{1}{d}I - |P\rangle\langle P|$, where P is the projector on the maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$. Therefore, in $d \otimes d$ dimensions the operator $T_{d\otimes d} = U^{\dagger}_{d\otimes d}W_{d\otimes d}U_{d\otimes d}$ detects $\chi_{d\otimes d}$ as a state which is not absolutely separable.

Entanglement creation using Quantum Gates: Consider a case in $d_1 \otimes d_2$ dimensions where $d_1 \neq d_2$. Consider the two qudit hybrid quantum gate U_H acting on $d_1 \otimes d_2$ dimensions, whose action is defined by

$$U_H|m\rangle \otimes |n\rangle = |m\rangle \otimes |m-n\rangle, \tag{10}$$

with $m \in \mathbb{Z}_{d_1}, n \in \mathbb{Z}_{d_2}$. Let us take the initial mixed separable state

$$\chi_{d_1 \otimes d_2} = \frac{1}{4}\chi_x + \frac{3}{4}\chi_y, \tag{11}$$

where χ_x is a projector on $\frac{1}{\sqrt{2}}(|0, d_2 - 1\rangle + |1, d_2 - 1\rangle)$, and χ_y a projector on $|d_1 - 1, d_2 - 1\rangle$.
$T_{d_1 \otimes d_2} = U_H^\dagger W_{d_1 \otimes d_2} U_H$ detects $\chi_{d_1 \otimes d_2}$ as a state which is not absolutely separable.

Effect of Global unitary operation on Conditional von Neumann entropy

Effect of Global unitary operation on Conditional von Neumann entropy

- The conditional von Neumann entropy is a distinctive feature of quantum states providing a means to validate a state's efficacy in quantum dense coding and state merging.
- The von Neumann entropy of a system ρ_{AB} with two subsystems A and B is denoted by $S(\rho_{AB})$. The conditional von Neumann entropy for ρ_{AB} entropy is defined as $S(\rho_{AB}) - S(\rho_A)$, where $S(\rho_A)$ denotes the von Neumann entropy of the subsystem A.
- The set of states whose conditional von Neumann entropy remains non-negative under any global unitary operations is denoted by **ACVENN**$= \{\sigma_{ac} :$ S$(U\sigma_{ac}U^{\dagger})$ - S$[(U\sigma_{ac}U^{\dagger})_A] \geq 0, \forall U\}$.

Characterization

Theorem
A state $\sigma_{ac} \in$ **ACVENN** *iff* $S(\sigma_{ac}) \geq 1$.

Observations

One may quickly note the following observations,

- Any pure separable state has a non-negative conditional entropy and can be brought by some unitary to a maximally entangled state which now possesses a negative conditional entropy and thus pure separable states can never belong to our desired class. Pure entangled states itself have a negative conditional entropy. Therefore, pure states are not eligible members of **ACVENN**.
- The maximally mixed state is a member of **ACVENN**.However, the maximally mixed state only constitutes a trivial example and we later find that the class contains some very non-trivial states.

Example:Bell diagonal state

- Bell-diagonal states can be expressed as, $\sigma_{bell} = \{\vec{0}, \vec{0}, T^b\}$, where $\vec{0}$ is the Bloch vector which is a null vector and the correlation matrix is $T^b = (c_1, c_2, c_3)$ with $-1 \leqslant c_i \leqslant 1$.
- The eigenvalues λ_1, λ_2, λ_3, λ_4 of Bell diagonal states are expressed as, $\lambda_1 = \frac{1}{4}(\chi - 2c_1)$, $\lambda_2 = \frac{1}{4}(\chi - 2c_2)$, $\lambda_3 = \frac{1}{4}(\chi - 2c_3)$, $\lambda_4 = \frac{1}{4}(2 - \chi)$, where $\chi = 1 + c_1 + c_2 + c_3$.
- As a result for the following condition we find the Bell diagonal states will belong to **ACVENN**

$$\log((\chi - 2c_2)(\chi - 2c_3)(2 - \chi)(\chi - 2c_1)) + c_1 \log(\frac{(\chi - 2c_2)(\chi - 2c_3)}{(2 - \chi)(\chi - 2c_1)}) + c_2 \log(\frac{(\chi - 2c_3)(\chi - 2c_1)}{(\chi - 2c_2)(2 - \chi)}) + c_3 \log(\frac{(\chi - 2c_2)(\chi - 2c_1)}{(\chi - 2c_3)(2 - \chi)}) \leq 4. \tag{12}$$

Example: Bell diagonal state

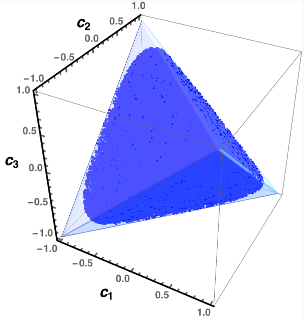

Figure 1: The von Neumann entropy of Bell diagonal state σ_{bell} against the parameter c_i

In figure 1, we consider an ensemble of 10^5 states within which the dark blue colour area at the centre of the octahedron determines the class of states $\in$**ACVENN**. The light blue areas at the corner indicate the states $\notin$ **ACVENN**.

Theorem
ACVENN *is a convex compact subset of* **Q***[set of all quantum states].*

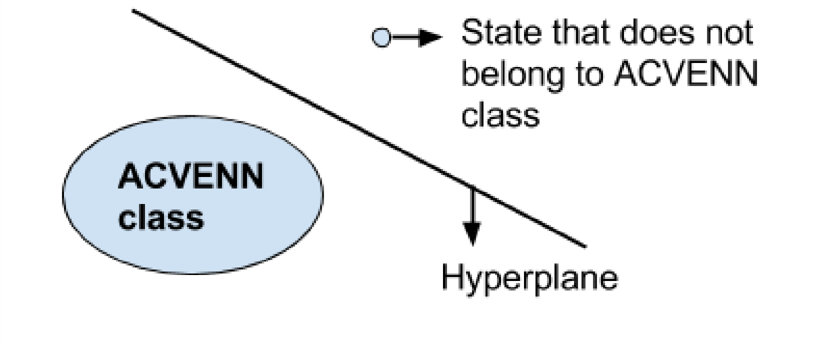

Figure 2: The set ACVENN is convex and compact.

Estimation of the size of ACVENN

Next we estimate the size of the **ACVENN** class by taking the maximum and minimum distance from the identity $(\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2})$. The distance measure we have used in this context is the Frobenius norm which is given by $\|X\| = \sqrt{Tr(X^{\dagger}X)}$.

- For any general $\widetilde{\varrho}$, distance from $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ is given by $\|\widetilde{\varrho} - \frac{\mathbb{I}}{4}\| = \sqrt{Tr((\widetilde{\varrho} - \frac{\mathbb{I}}{4})^{\dagger}(\widetilde{\varrho} - \frac{\mathbb{I}}{4}))}$, which on solving further results to $\sqrt{Tr(\widetilde{\varrho}^2) - \frac{1}{4}}$.
- To calculate the maximum distance we needed to maximise $\|\sigma - \frac{\mathbb{I}}{4}\|$, over all $\sigma \in$ **ACVENN**. Here we solve this problem numerically. After going through 2×10^5 **ACVENN** states the maximum distance we have is 0.645966 by the state whose eigen values were $\lambda_1 = 0.809161$, $\lambda_2 = 0.0521141$, $\lambda_3 = 0.0595448$, $\lambda_4 = 0.0791805$.

- To calculate the minimum distance we needed to minimise $\|\rho - \frac{\mathbb{I}}{4}\|$, over all $\rho \notin$ **ACVENN**. Going through 1×10^5 **non-ACVENN** states numerically, we attained the minimum distance as 0.507225. This is given by a state whose eigen values were $\lambda_1 = 0.00014347$, $\lambda_2 = 0.000551157$, $\lambda_3 = 0.436523$, $\lambda_4 = 0.562783$.

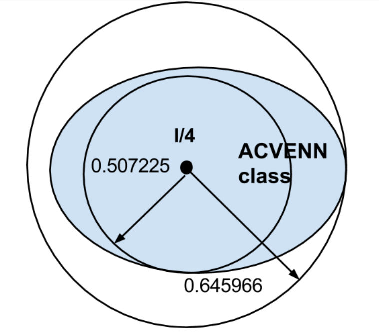

Figure 3: The figure depicts the approximate size of the **ACVENN** class

Application: Super dense coding

In particular, super dense coding capacity for a mixed state ρ_{AB} in $D(H_d \otimes H_d)$ is defined by

$$\mathcal{C}_{AB} = max\{\log_2 d, \log_2 d + S(\rho_B) - S(\rho_{AB})\}, \tag{13}$$

where, $\rho_B = tr_A[\rho_{AB}]$. $\mathcal{C}_{AB}$ is nothing but the amount of classical information that can be sent from system A to system B. Here we note that the expression $S(\rho_B) - S(\rho_{AB})$ can either be positive or negative.

Application: Super dense coding

For our first example let's consider a mixed separable state in $D(H_2 \otimes H_2)$ given by

$$\rho = \begin{vmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 1-a & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \tag{14}$$

The state does not provide any quantum advantage in super dense coding. However on application of the unitary operator

$$U_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}, \tag{15}$$

the state becomes useful for super dense coding.

Application: Quantum state merging

- In the information processing scenarios, it is important to ask this question: if an unknown quantum state is distributed over two systems say A and B, how much quantum communication is needed to transfer the full state to one system.
- This communication measures the *partial information* one system needs conditioned on it's prior information. Remarkably, this is given by the conditional entropy $S(A|B)$ (if it is from A to B) of the system.
- It is interesting to note that in principle this entropy can be positive ($S(A|B) > 0$), negative ($S(A|B) < 0$) and zero ($S(A|B) = 0$), where each have different meaning in the context of state merging.
- If the partial information is positive, its sender needs to communicate this number of quantum bits to the receiver; if zero it tells there is no need of such communication; if it is negative, the sender and receiver instead gain the corresponding potential for future quantum communication.

Application: Quantum state merging

Theorem
For a given spectrum of density matrix the minimum state merging cost will be achieved at the Bell diagonal states .

Extension of the problem to Bell-CHSH local states

Extension of the problem to Bell-CHSH local states

- A state is considered to be non-local if it violates some Bell inequality.
- One very important Bell's inequality is the Bell-CHSH inequality, expressed as

$$Tr(B_{CHSH}\rho) \leq 2 \tag{16}$$

  where $B_{CHSH} = \hat{a}.\overrightarrow{\sigma} \otimes (\hat{b} + \hat{b}').\overrightarrow{\sigma} + \hat{a}'.\overrightarrow{\sigma} \otimes (\hat{b} - \hat{b}').\overrightarrow{\sigma}$ The above inequality can also be translated in the form $Tr(B^{W}_{CHSH}\rho) \geq 0$.
- A state ρ_l for which $Tr(B_{CHSH}\rho_l) \leq 2, \forall B_{CHSH}$ will be called as Bell-CHSH Local. A state σ_{al} will be called as absolutely Bell-CHSH local if $Tr(B_{CHSH}U\sigma_{al}U^{\dagger}) \leq 2, \forall B_{CHSH}, \forall U$.

We denote by **L**, the set of all states which do not violate the Bell-CHSH inequality.We denote by **AL** as the set containing states which do not violate the Bell-CHSH inequality under any global unitary operation.

The Problem

(i) Can we characterize **AL** ?

(ii) Can we put some restrictions on the spectrum of a density matrix for it to be an Absolute Bell-CHSH local state?

(iii) What can be the corresponding restrictions on the bloch parameters of the state?

Bell-CHSH Local from spectrum

- A density matrix living in $\mathbf{B}(\mathbf{C}^2 \otimes \mathbf{C}^2)$ can be expressed in the Hilbert-Schmidt basis as ,

$$\sigma = \frac{1}{4}[I \otimes I + \vec{u}.\vec{s} \otimes I + I \otimes \vec{v}.\vec{s} + \Sigma_{i,j=1}^{3} t_{ij} s_i \otimes s_j] \qquad (17)$$

  Here, $\vec{u}, \vec{v}$ are the local Bloch vectors and $t_{ij} = Tr[\sigma(s_i \otimes s_j)]$, s_i are the Pauli matrices.

- In a much celebrated work [Horodecki et. al 1995], the Horodecki family derived a necessary and sufficient condition to verify whether a state is local with respect to the Bell-CHSH inequality, based on the parameters of the state.
- The condition was based on the value of a function $M(\rho_l) = \lambda_{max1} + \lambda_{max2}; \lambda_{max1}, \lambda_{max2}$ being the maximum two eigen values of $Y = T^t T$ where $T = [t_{ij}]$ is the correlation matrix of ρ_l and t denotes transposition.

- It was then stated that a state ρ_l is Bell-CHSH local iff $M(\rho_l) \leq 1$. The maximal Bell-CHSH violation for any state χ was then showed to be $2\sqrt{M(\chi)}$.
- In this light, the *absolutely Bell-CHSH local* set can also be defined as $\mathbf{AL} = \{\sigma_{al} : M(U\sigma_{al}U^\dagger) \leq 1, \forall U\}$.

Theorem
A state σ is absolutely Bell-CHSH local if and only if $(2a_1 + 2a_2 - 1)^2 + (2a_1 + 2a_3 - 1)^2 \leq 1$, where a_1, a_2, a_3 are the highest three eigenvalues of σ in a decreasing order.

Characterization in terms of Bloch parameters

Theorem
A Bell diagonal state is absolutely Bell-CHSH local iff $Max(t_{11}^2 + t_{22}^2, t_{11}^2 + t_{33}^2, t_{22}^2 + t_{33}^2) \leq 1$, *where* t_{ii} *are the diagonal elements of the correlation matrix of the Bell diagonal state.*

In $2 \otimes 2$ system, any unitary operator U can be decomposed as : $U = U_1 \otimes U_2 U_{\theta_1\theta_2\theta_3} U_3 \otimes U_4$, where $U_i s$ are the local unitary operators and $U_{\theta_1\theta_2\theta_3}$ the basic nonlocal unitary operator. For states which are diagonal in the computational basis, the only non-zero parameters in its Hilbert-Schmidt representation are u_3, v_3, t_{33}. As a result we have the following result,

Theorem

A state σ_{comp} diagonal in the computational basis is absolutely Bell-CHSH local iff
$Max[\underset{\theta_1,\theta_2,\theta_3}{Max}(t_{33}^2 + c_{11}^2), \underset{\theta_1,\theta_2,\theta_3}{Max}(t_{33}^2 + c_{22}^2), \underset{\theta_1,\theta_2,\theta_3}{Max}(c_{11}^2 + c_{22}^2)] \leq 1$, *where*
$c_{11} = u_3 cos\theta_2 sin\theta_1 - v_3 sin\theta_2 cos\theta_1, c_{22} = v_3 cos\theta_2 sin\theta_1 - u_3 sin\theta_2 cos\theta_1$.
($\theta_i s$ being the parameters of the basic non-local operator)

For any arbitrary density matrix σ it is in general difficult to arrive at the necessary and sufficient conditions in a closed form. However, we can arrive at a sufficient condition as given below,

Theorem
If for any density matrix σ, $\underset{\theta_1,\theta_2,\theta_3}{Max}\, Tr((T'_\sigma)^t T'_\sigma) \leq 1$ then $\sigma \in$ **AL**

Comparison with purity and the Absolutely Bell-CHSH Local Ball

Let us assume that the eigenvalues of the density matrix are a_1, a_2, a_3, a_4 in descending order. The condition for a state to be *absolutely Bell-CHSH local* can also be re-framed as $(a_1 - a_4)^2 + (a_2 - a_3)^2 \leq 1/2$. The problem of finding the maximum purity beyond which there cannot be any *absolutely Bell-CHSH local* state can now be posed as :

$$\begin{aligned} & Maximize(a_1^2 + a_2^2 + a_3^2 + a_4^2), \\ & subject \;\; to : \\ & (i)(a_1 - a_4)^2 + (a_2 - a_3)^2 \leq 1/2 \\ & (ii)a_1 + a_2 + a_3 + a_4 = 1 \\ & (iii)1 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0 \end{aligned} \tag{18}$$

A numerical computation yields the answer as $\frac{5}{8}$. The solution signifies that if the purity of a density matrix is greater than $\frac{5}{8}$, it cannot be a *absolutely Bell-CHSH local* state.

The converse question of finding the minimum purity below which there cannot be any *non-absolutely Bell-CHSH local* state can be posed as :

$$\begin{aligned}
&Minimize(a_1^2 + a_2^2 + a_3^2 + a_4^2),\\
&subject \;\; to:\\
&(i)(a_1 - a_4)^2 + (a_2 - a_3)^2 > 1/2\\
&(ii)a_1 + a_2 + a_3 + a_4 = 1\\
&(iii)1 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0 \qquad (19)
\end{aligned}$$

Numerically, the answer is obtained as $\frac{1}{2}$. This signifies that all states with a purity $\leq \frac{1}{2}$ is *absolutely Bell-CHSH local*.

- The above result can also be framed in terms of the Frobenius norm $\|A\|^2 = Tr(A^\dagger A)$.
- Specifically, $Tr(\rho^2) \leq 1/2 \implies \|\rho - I/4\| \leq 1/2$, which is the *absolutely Bell-CHSH local* ball. Every state within this ball is *absolutely Bell-CHSH local*. ($I/4$ is the maximally mixed state).
- On the other hand $Tr(\rho^2) > 5/8 \implies \|\rho - I/4\| > \sqrt{3}/2\sqrt{2}$. Thus any state outside this ball cannot be *absolutely Bell-CHSH local*.

Illustrations

- *Absolutely Separable states* - Any absolutely separable state will be *absolutely Bell-CHSH local*. This is because , absolutely separable states preserve their separability under global unitary operation. In set theoretic language, if we denote the set containing the absolutely separable states by **AS**, then **AS** forms a proper subset of **AL**.
- If we take a state diagonal in computational basis, say $\sigma_{mix} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$, then the state is *absolutely Bell-CHSH local* as, $(1/2-0)^2 + (1/2-0)^2 \leq 1/2$.
  One can verify this from the perspective of bloch parameters. The only bloch parameter in this case is $t_{33} = 1$. Using the result obtained for states diagonal in computational basis , one sees that the state $\in$ **AL**.

- *Werner state*- The Werner state is given as, $\sigma_{wer} = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}I(|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}})$. The state is absolutely separable for $p \leq 1/3$, hence *absolutely Bell-CHSH local* there. The eigenvalues are $\{(1+3p)/4, (1-p)/4, (1-p)/4, (1-p)/4\}$. For $p \leq 1/\sqrt{2}$ it is *absolutely Bell-CHSH local*. One may note that, the Werner state is a Bell diagonal state and it is *Bell-CHSH local* upto $1/\sqrt{2}$.
  This characterization can also be done in terms of the bloch parameters. As for the Werner state $t_{11} = t_{22} = t_{33} = -p$, $Max(t_{11}^2 + t_{22}^2, t_{11}^2 + t_{33}^2, t_{33}^2 + t_{22}^2) = 2p^2$. Therefore, for $p \leq 1/\sqrt{2}$, it is *absolutely Bell-CHSH local*.Once again this reiterates the fact whenever a Bell diagonal state is *Bell-CHSH local* it is *absolutely Bell-CHSH local*.

Figure: AL , AS , S

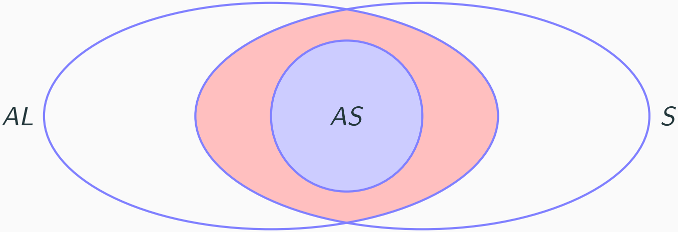

Figure 4: Relation between **AL**, **AS**, **S**. The red shaded region indicates some states from **S** − **AS** which are in **AL**.

Relation between AS, ACVENN and AL

I. AS vs ACVENN

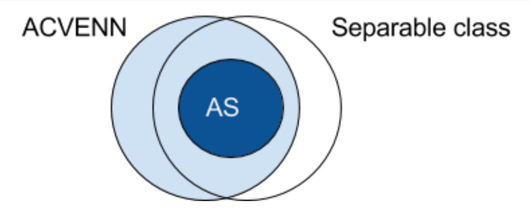

Figure 5: The figure depicts the relation between **ACVENN**, **AS** and separable class of states

AS forms a proper subset of **ACVENN**.

Relation between AS, ACVENN and AL

II. AL vs ACVENN

- The Werner states are absolutely local for the visibility factor $p \leq 1/\sqrt{2}$, and they belong to **ACVENN** for $p \leq 0.7476$.
- Therefore, the *absolutely Bell-CHSH local* Werner states form a subset of the **ACVENN** class.
- This is an interesting result as that would mean, there are states that violate *Bell-CHSH* inequality, and still under any unitary cannot be improved to a state with negative conditional entropy.

However, it is difficult to comment in general on the relation between **AL** and **ACVENN**.

Future Directions

Future Directions

- A possible direction of research is to probe the conditions for the existence of Absolute Local hidden state models.
- One can also explore the **ACVENN** in higher dimensions.

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