# Tomographic Image Reconstruction in Noisy and Limited Data Settings.

Syed Tabish Abbas

International Institute of Information Technology, Hyderabad

*syed.abbas@research.iiit.ac.in*

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# Problem Statement and Contributions

- We investigate the tomographic reconstruction under 2 scenarios
    - Noisy Data case.
    - Limited Data case.

  and consider the following questions.

- In linear Radon transform, does reconstruction lattice play a role in quality of reconstructed image?
- How to reconstruct an image under limited view circular Radon Transform: the Circular arc Radon transform?
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# Reconstruction onto Hexagonal Lattices.

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# Positron Emission Tomography(PET)

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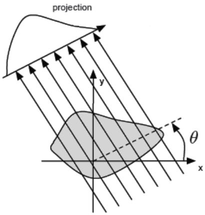


Figure: Forward Projection.

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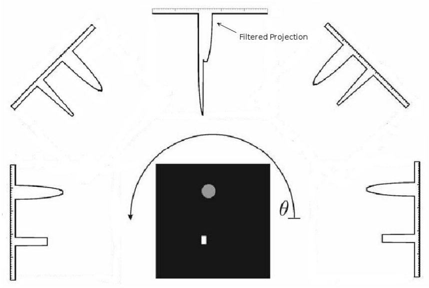
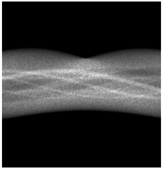
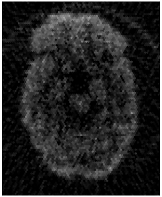


# Handling Noisy Data





Reconstructed Image

1

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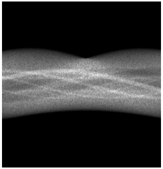
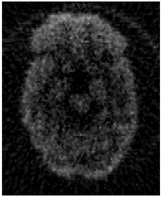
- More sophisticated methods, like algebraic inversion[1], Statistical inversion[2], etc. have also been proposed





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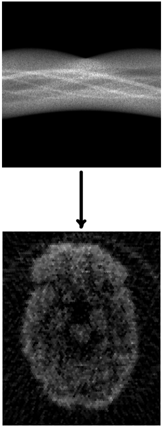

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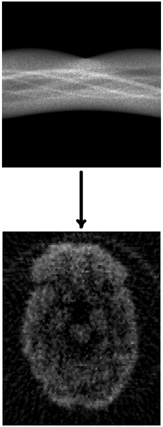
- Other methods, follow a two step process of reconstruction followed by denoising[3].
- *Reconstruction onto a different lattice has received very little attention.*



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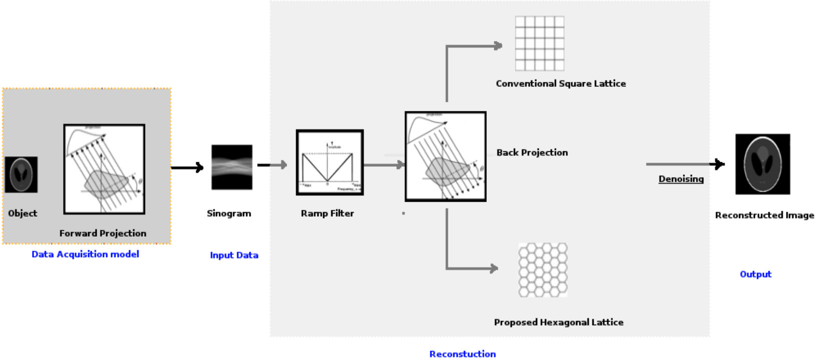
# Our Pipeline

We propose a 2 step reconstruction process onto Hexagonal lattice:



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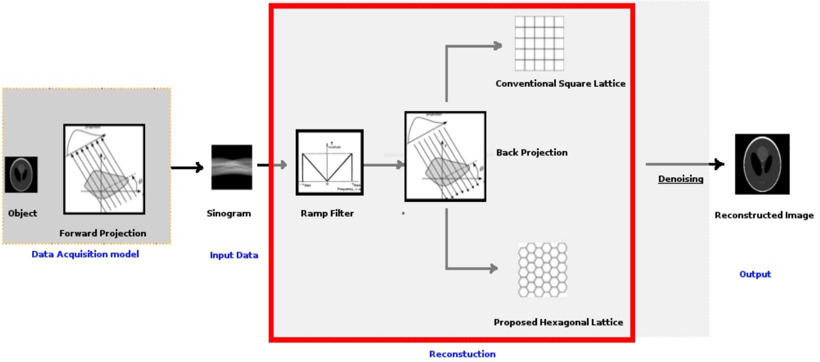
$\rightarrow$ *Step 1:* Noisy Reconstruction using Filtered Back Projection.



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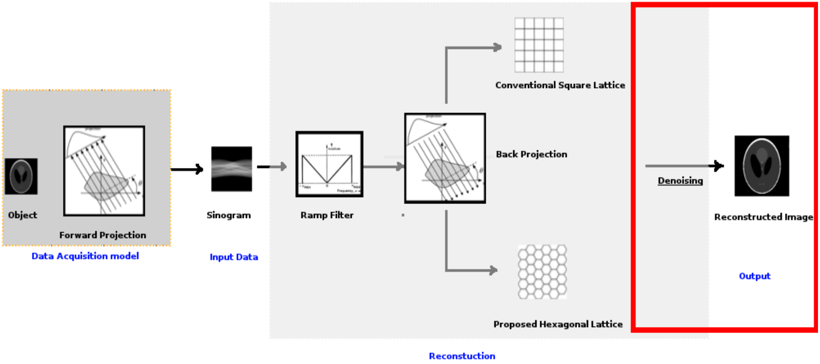

# Tiling of Euclidean Plane

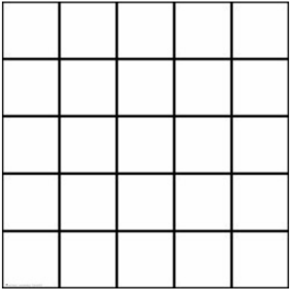

Figure: Square Tiling of Euclidean Plane

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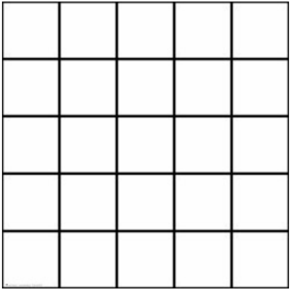

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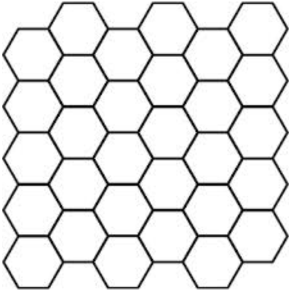

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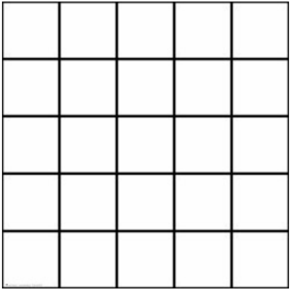

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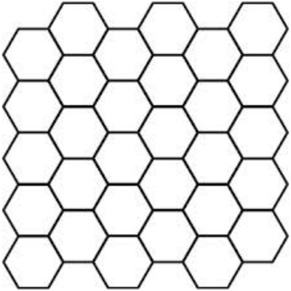

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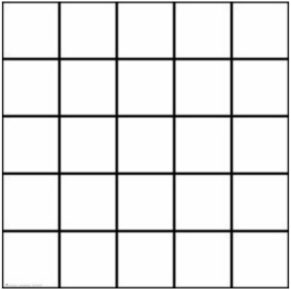

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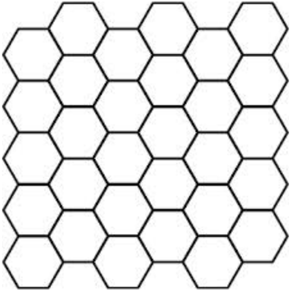

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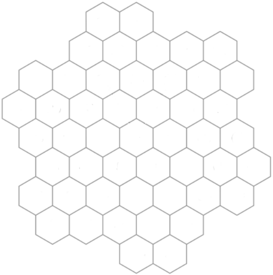
- ✗ Irrational Coordinates

# Addressing Hexagonal Lattices



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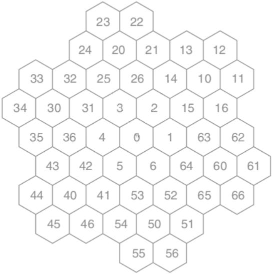
# Addressing Hexagonal Lattices

→ Use base 7 indices

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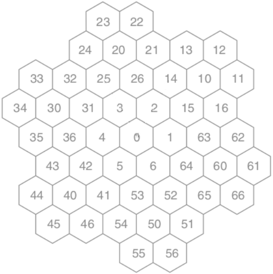


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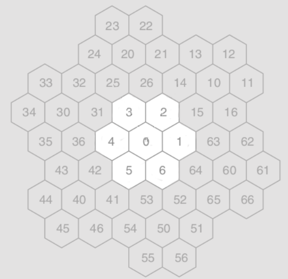


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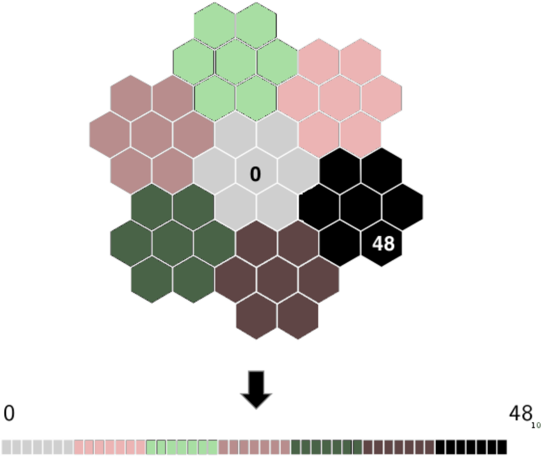


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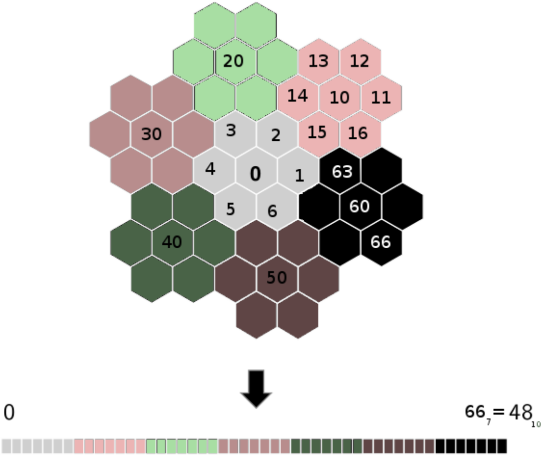


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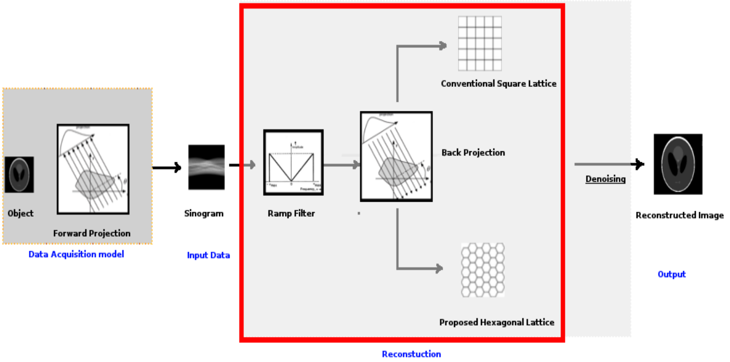
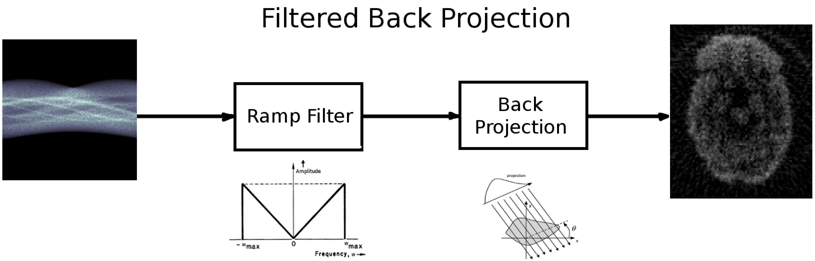


# Filtered Back-Projection (FBP)



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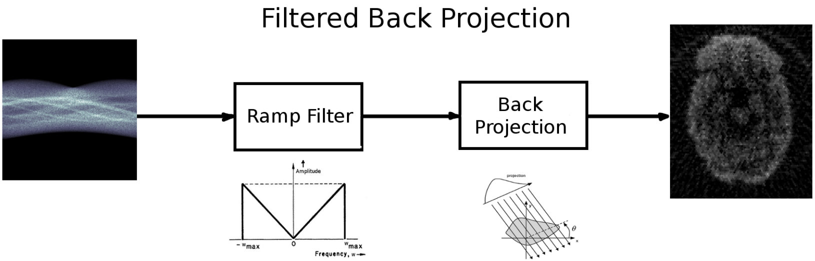


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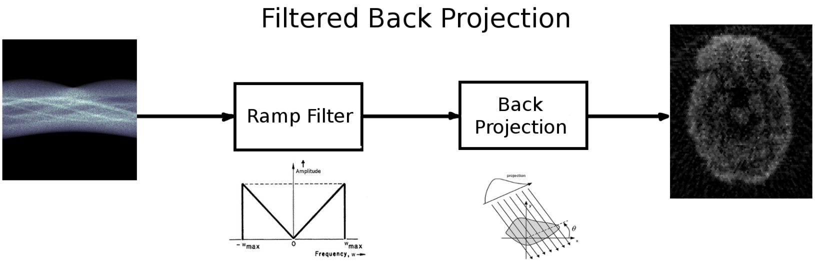


- ✗ Image reconstruction (especially in nuclear modalities) is very noisy.
- ✓ Back-projection (and also other reconstruction methods) allows a choice of reconstruction lattice.

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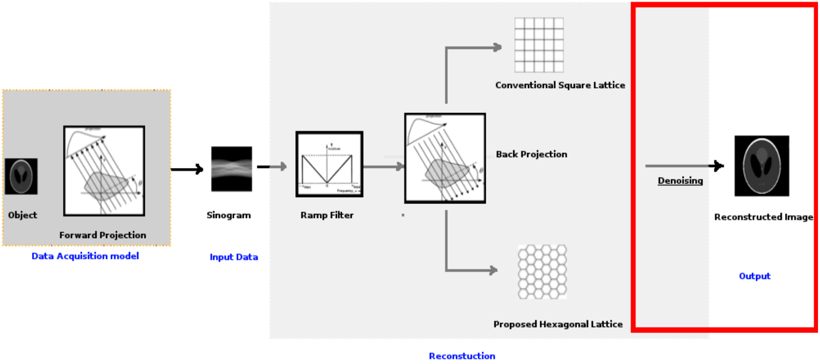

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- → Dictionary is learned by solving the following optimization problem.

$$\min_{D \in \mathcal{C}, \alpha \in \mathcal{R}^{k \times n}} \frac{1}{2} \parallel \mathbf{X} - \mathbf{D}\alpha \parallel_F^2 + \lambda \parallel \alpha \parallel_{1,1}$$

$$\mathcal{C} = \{\mathbf{D} \in \mathcal{R}^{m \times k} s.t \forall j = 1, ...k, ||d_j^T||_2 \leq 1\}$$

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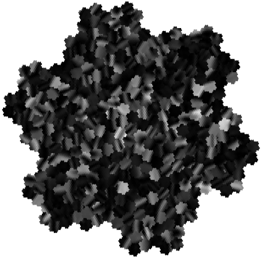

Figure: Sample Dictionary atoms

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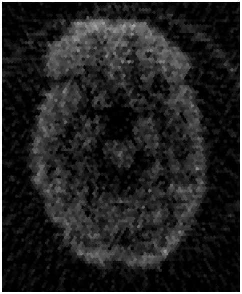

Noisy Image

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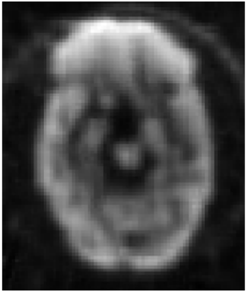

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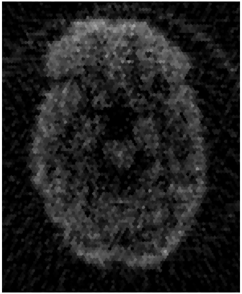

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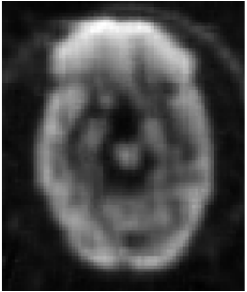

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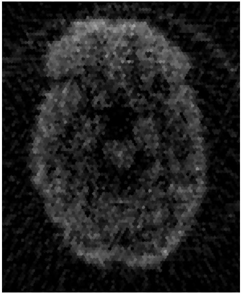

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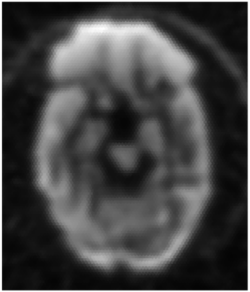

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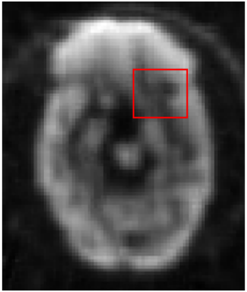

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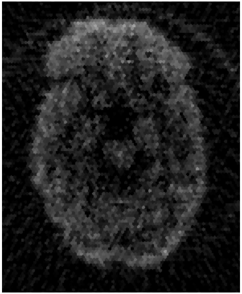

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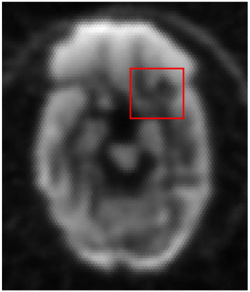

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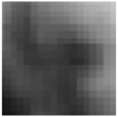

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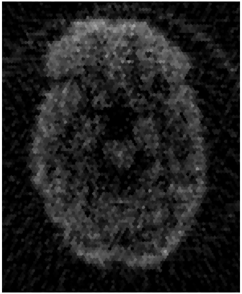

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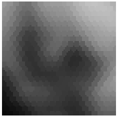

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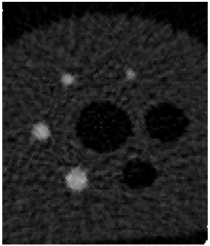

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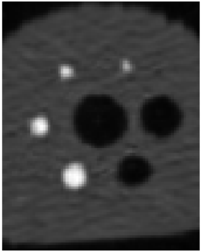

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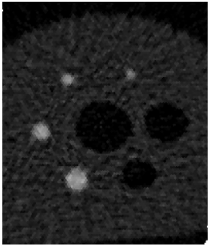

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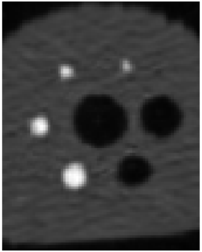

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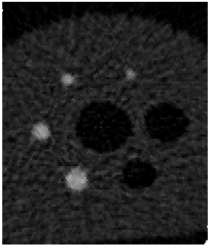

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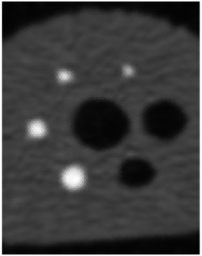

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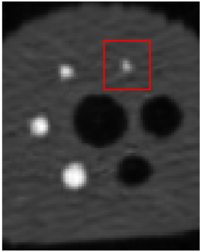

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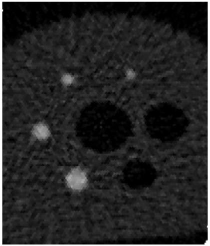

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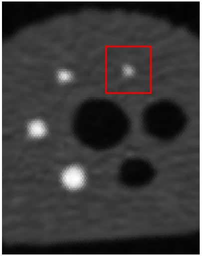
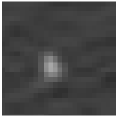
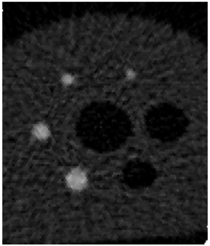


Hexagonal lattice

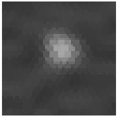
# Qualitative Results



Square lattice



Noisy Image



Hexagonal lattice

# Quantitative Results

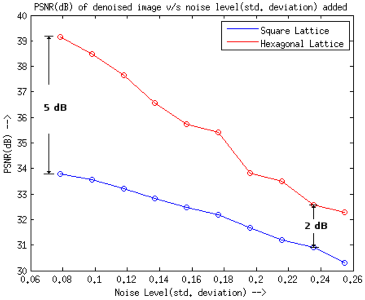



Figure: PSNR Comparison

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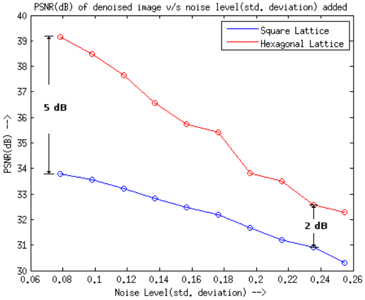


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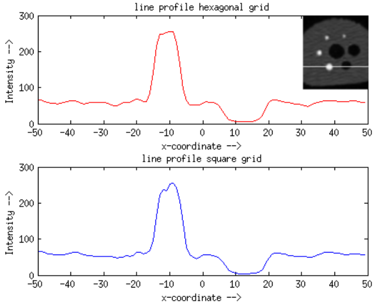


Figure: Line Profile

# Summary & Future Work

- ✓ We Proposed that the change of lattice can improve the reconstruction quality of PET images.
- ✓ The change in lattice improves both the quality and fidelity of the final denoised image.
- ✓ Include the noise model in the denoising step.
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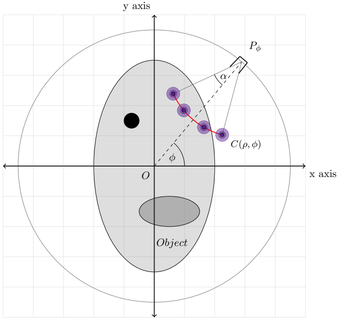
# Reconstruction In Limited View Scenario

How to reconstruct an image under limited view circular Radon Transform: the Circular arc Radon transform?

# Imaging setup

- Type: Photoacoustic type sensors where source of excitement is EM waves and measurement is acoustic waves.

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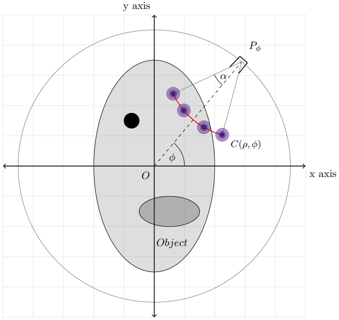


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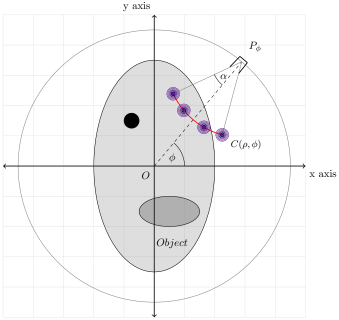


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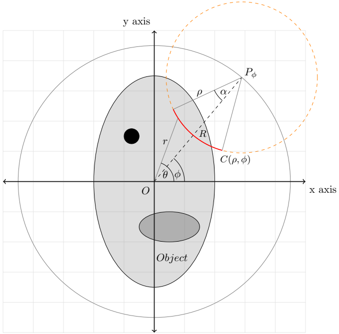


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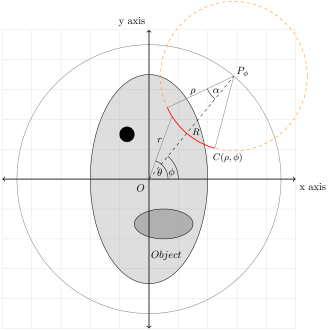
# Mathematial Model

We define Circular arc Radon(CAR) Transform $g^\alpha$ of a function $f$ as follows

$$g^\alpha(\rho,\phi) = \int_{A_\alpha(\rho,\phi)} f(r,\theta)\,\mathrm{d}s \quad (1)$$

where is $\alpha$ is the view angle and $s$ is the arc length measure.

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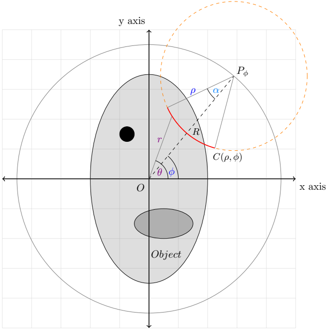


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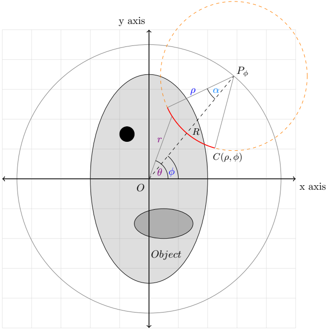


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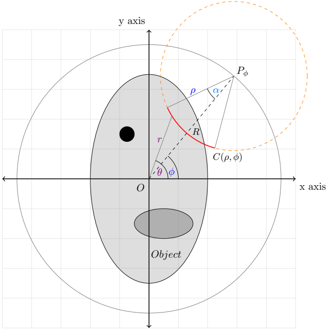


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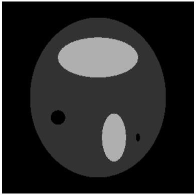
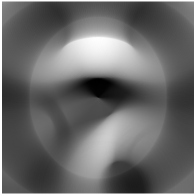
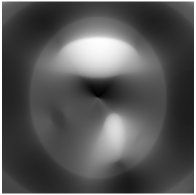


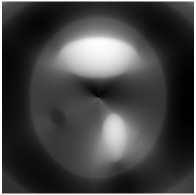
(a) Original Phantom



(b) $\alpha = 5$



(c) $\alpha = 17$



(d) $\alpha = 21$

Examples of image reconstructions using a näive Backprojection Algorithm

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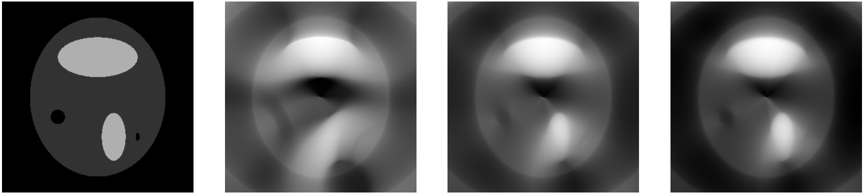

(a) Original Phantom (b) $\alpha = 5$ (c) $\alpha = 17$ (d) $\alpha = 21$

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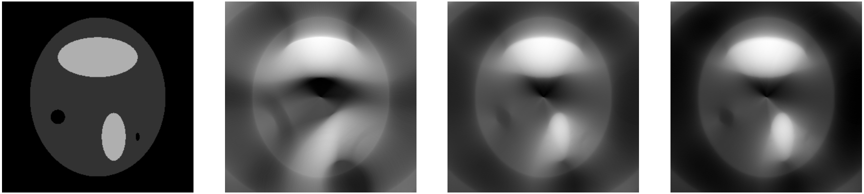

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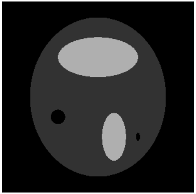

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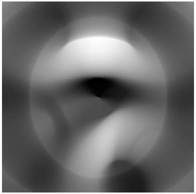

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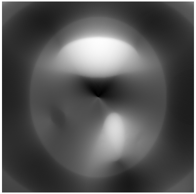

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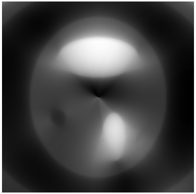

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Examples of image reconstructions using a näive Backprojection Algorithm

- The BP based algotithm is an approximate inversion and leads to lot of artifacts as well as blurring.
- Due the form of transform, it is non-trivial to derive the exact form of the filter.
- To improve the quality of reconstruction, we adopt a **Fourier series** based solution.

$$\boxed{g^{\alpha}(\rho,\phi) = \int\limits_{A_\alpha(\rho,\phi)} f(r,\theta)\,\mathrm{d}s}$$

Since both $f$, $g$ are $2\pi$ periodic in angular variable, we may expand them into their Fourier series such that,

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then,

$$\sum_{n=-\infty}^{\infty} g_n^{\alpha}(\rho)\, e^{in\phi} = \sum_{n=-\infty}^{\infty} \int\limits_{A_{\alpha}(\rho,\phi)} f_n(r) e^{in\theta} \mathrm{d}\theta.$$

# CAR Transform: Fourier Series based analysis

On Simplifying and equating the Fourier coefficients, the equation reduces to

$$\boxed{g_n^{\alpha}(\rho) = \int_{R-\sqrt{R^2+\rho^2-2\rho R\cos\alpha}}^{\rho} \frac{K_n(\rho,u)}{\sqrt{\rho-u}} F_n(u)\mathrm{d}u}$$

where

$$F_n(u) = f_n(R-u)$$

and

$$K_n(\rho,u) = \frac{2\rho(R-u)T_n\left[\frac{(R-u)^2+R^2-\rho^2}{2R(R-u)}\right]}{\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}}. \quad (2)$$

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- ☺ A direct numerical solution of the equation does not require closed form solution.

# Discrete CAR Transform

$$g_n^{\alpha}(\rho) = \int\limits_{R-\sqrt{R^2+\rho^2-2\rho R\cos\alpha}}^{\rho} \frac{K_n(\rho,u)}{\sqrt{\rho-u}} F_n(u)\mathrm{d}u.$$

$$g_n^{\alpha}(\rho_k) = \sum_{q=1}^{k} \int\limits_{\rho_{q-1}}^{\rho_q} \frac{F_n(u)K_n(\rho,u)}{\sqrt{\rho-u}}\mathrm{d}u.$$

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Approximating the integrand as a linear function over each interval $[\rho_{q-1}, \rho_q]$, and integrating we get

$$g_n(\rho_k) = \sqrt{h} \left\{ \sum_{q=l}^{k} b_{kq} K_n(\rho_k, \rho_q) F_n(\rho_q) \right\}$$

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$$b_{kq} = \begin{cases} \frac{4}{3}\{(k-q+1)^{\frac{3}{2}} + \frac{4}{3}(k-q)^{\frac{3}{2}} + 2(k-q)^{\frac{1}{2}} & q = l \\ \frac{4}{3}\left((k-q+1)^{\frac{3}{2}} - 2(k-q)^{\frac{3}{2}} + (k-q-1)^{\frac{3}{2}}\right) & q = l+1, \ldots k-1. \\ \frac{4}{3} & q = k. \end{cases}$$

and $l = \max\left(0, \left\lfloor R - \sqrt{R^2 + \rho_k^2 - 2\rho_k R \cos\alpha} \right\rfloor\right)$ where $\lfloor x \rfloor$ is the greatest integer less than equal to $x$.

# Discrete CAR Transform

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- Diagonal entries of $B_n$, $b_{ii} = \frac{4}{3}\sqrt{h} \neq 0$, hence the matrix is invertible.

# Numerical inversion of CAR Transform.

$$g_n^\alpha = B_n F_n$$

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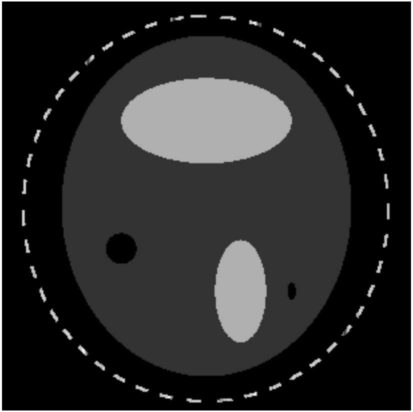
We use a Truncated SVD based $r$-rank inverse ($r < M$) such that,

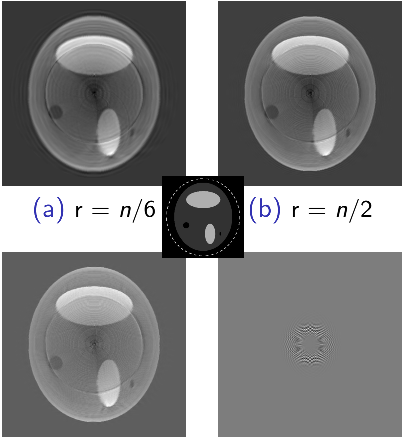
$$F_n \approx B_{n,r}^{-1} \; g_n^{\alpha}$$

# Experiments and Results: Effect of Rank



original phantom ($f$) used in experiments.

- Full rank inversion is expected to be unstable.



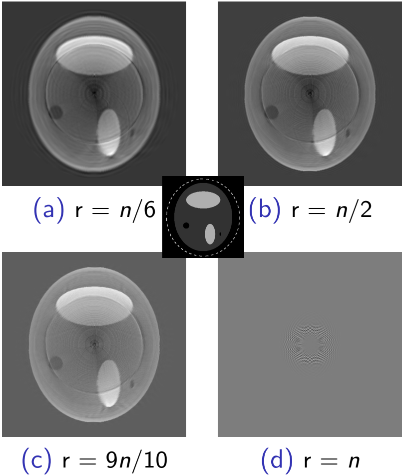
(a) r $= n/6$ (b) r $= n/2$

(c) r $= 9n/10$ (d) r $= n$

Effect of rank $r$ of matrix $B_{n,r}$ on the reconstruction quality. n $= 300$

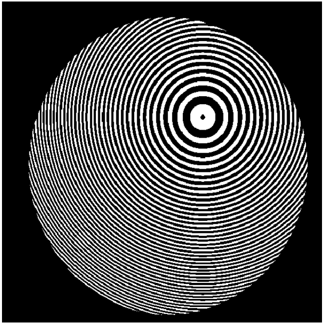
# Experiments and Results: Effect of Rank

- Full rank inversion is expected to be unstable.
- If the rank $r$ is set to be too low reconstructed image is expected to have ringing artifacts.




Effect of rank $r$ of matrix $B_{n,r}$ on the reconstruction quality. $n = 300$

# Experiments and Results: Choosing Rank



original phantom ($f$) used in experiments.

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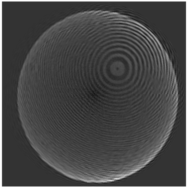
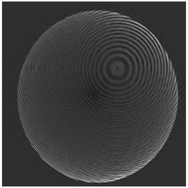
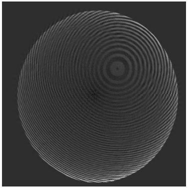


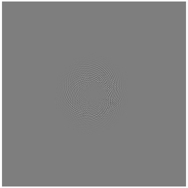
(a) r $= n/3$



(b) r $= n/2$

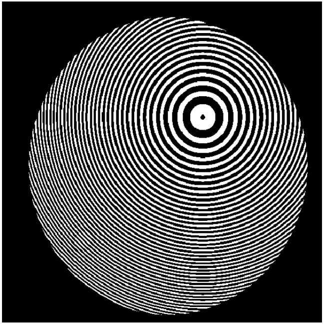


(c) r $= 9n/10$



(d) r $= n$

Effect of rank $r$ of matrix $B_{n,r}$ on the reconstruction quality. n $= 300$



original phantom ($f$) used in experiments.

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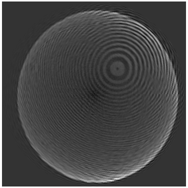

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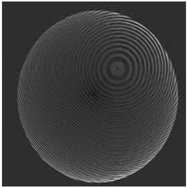

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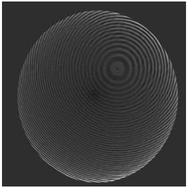

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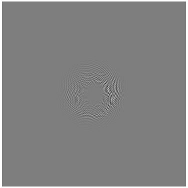

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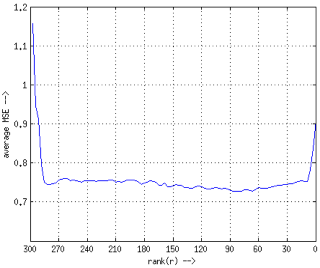
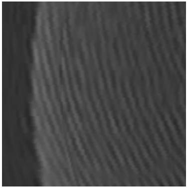
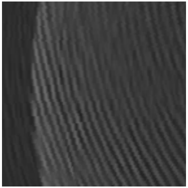



Plot of Mean Square Error as a function of rank $r$.

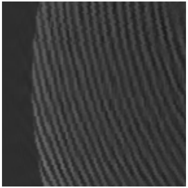
# Experiments and Results: Choosing Rank



(a) r $= n/3$



(b) r $= n/2$



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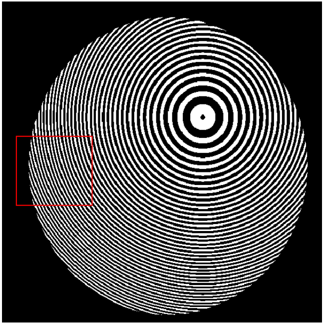
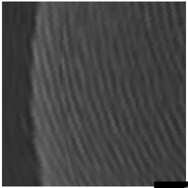


original phantom ($f$) used in experiments where region to be zoomed is shown in red.

# Experiments and Results: Choosing Rank



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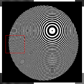

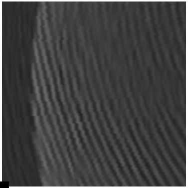

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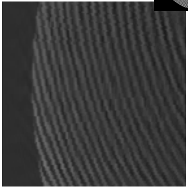


(c) r $= 9n/10$

(d) r $= n$

Effect of rank $r$ of matrix $B_{n,r}$ on the reconstruction quality. n $= 300$

Based on our experiments, as a rule of thumb, dropping highest 10% of singular values gives a fairly stable reconstruction.

# Reconstruction In Limited View Scenario

How to remove the artifacts which arise in the Circular arc Radon transform due to the limited view?

# Reconstruction of Singularities under Radon Transforms.

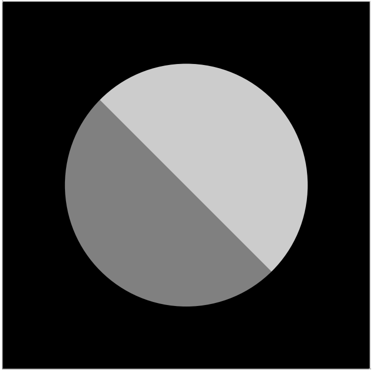
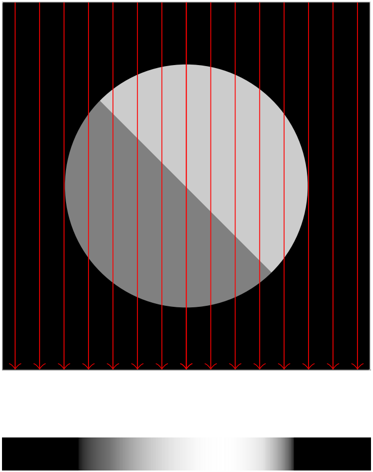


Figure: Image with visualization of projection value along direction shown.



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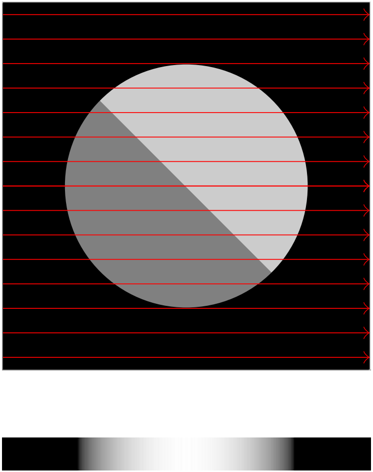

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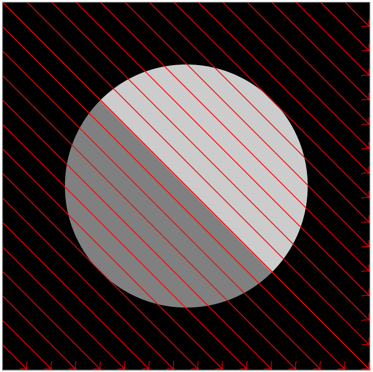

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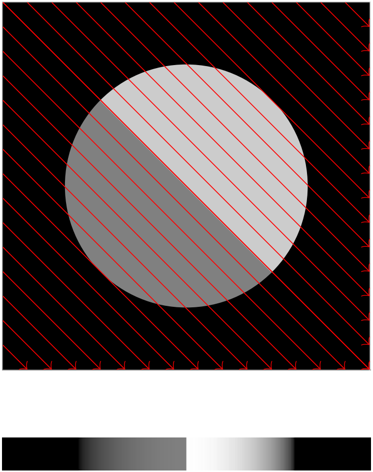

Figure: Image with visualization of projection value along direction shown.

Let $\mathcal{C}$ be the set of curves, along which we measure projections. Then for an edge to be visible there must be at least one element in the interior of set $\mathcal{C}$, tangential to the edge

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- Due to limited view **not all edges are visible**, in the sense of meeting tangency criterion w.r.t set $\mathcal{C}$.

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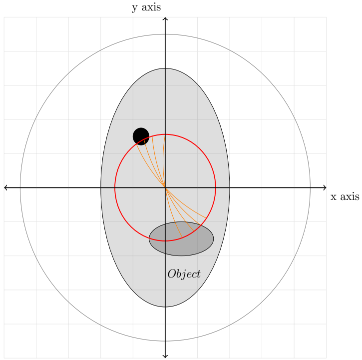


Figure: A sharp circular artifact is observed due to discontinuity in angular, as well as radial direction

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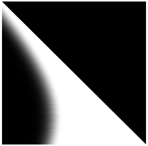
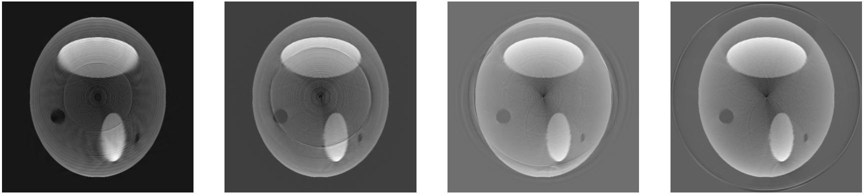
Visualization of structure of unmodified original matrix $B_n$.



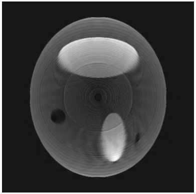
Visualization of structure of modified matrix $B_n$ for artifact suppression.

# Experiments and Results: Artifact Reduction.

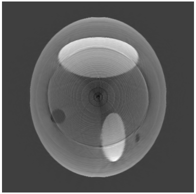


(a) $\alpha = 21$ (b) $\alpha = 31$ (c) $\alpha = 46$ (d) $\alpha = 76$

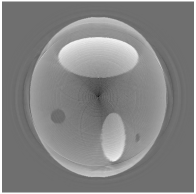
# Experiments and Results: Artifact Reduction.



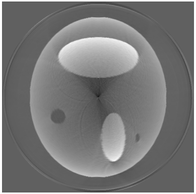
(a) $\alpha = 21$



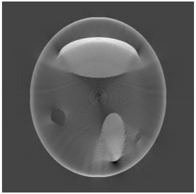
(b) $\alpha = 31$



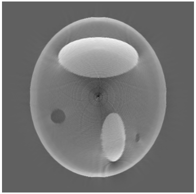
(c) $\alpha = 46$



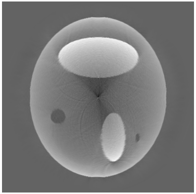
(d) $\alpha = 76$



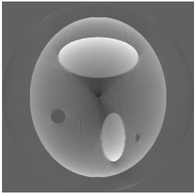
(e) $\alpha = 21$



(f) $\alpha = 31$



(g) $\alpha = 46$



(h) $\alpha = 76$

Reconstructed images corresponding to different $\alpha$ before (row 1), and after artifact suppression (row 2).

# Experiments and Results: Artifact Reduction

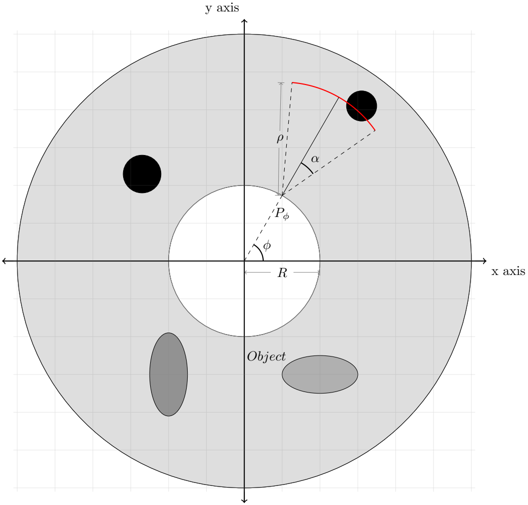
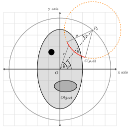



Figure: Setup with support outside the acquisition circle

# Experiments and Results: Artifact Reduction

$$g_n^{\alpha}(\rho) = \int_{\sqrt{R^2+\rho^2+2\rho R\cos\alpha}\;-R}^{\rho} \frac{K_n(\rho,u)}{\sqrt{\rho-u}} F_n(u)\mathrm{d}u$$



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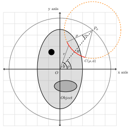
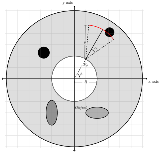



$$\downarrow$$




$$g_n^{\alpha}(\rho) = \int_{R-\sqrt{R^2+\rho^2-2\rho R\cos\alpha}}^{\rho} \frac{K_n(\rho,u)}{\sqrt{\rho-u}} F_n(u)\,\mathrm{d}u$$

# Experiments and Results: Artifact Reduction

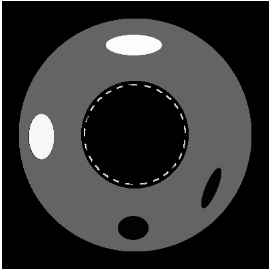
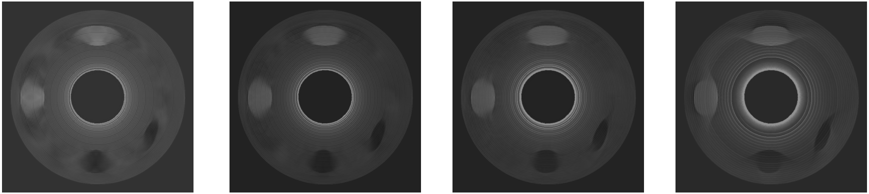


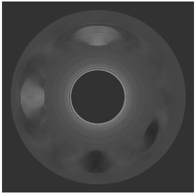
Figure: Phantom with support outside the acquisition circle.



(a) $\alpha = 21$ (b) $\alpha = 31$ (c) $\alpha = 46$ (d) $\alpha = 76$

Figure: Reconstructed images corresponding to different $\alpha$ before (row 1), and after artifact

# Experiments and Results: Artifact Reduction



(a) $\alpha = 21$

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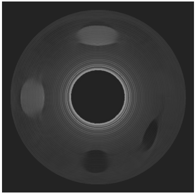

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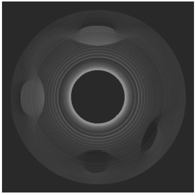

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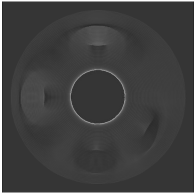

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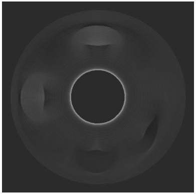
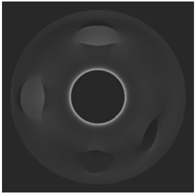
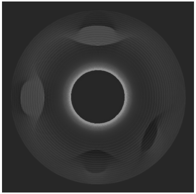


(f) $\alpha = 31$



(g) $\alpha = 46$



(h) $\alpha = 76$

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# Summary & Future Work.

✓ We Proposed a method of numerical inversion of circular arc Radon transform, a limited view generalization of circular Radon transform.

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- ✓ We Proposed a method of numerical inversion of circular arc Radon transform, a limited view generalization of circular Radon transform.
- ✓ We also proposed a strategy to reduce the artifacts which arise in the image due to limited view.

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✓ Provide a rigorous mathematical justification of the artifacts.

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- ✓ Provide a rigorous mathematical justification of the artifacts.
- ✓ Derive a closed form solution of the Volterra integral equation arising in the transform.

# Related Publications

- *PET Image Reconstruction And Denoising On Hexagonal Lattices.*
  **Syed T. A.** and Sivaswamy J.
  International Conference on Image Processing(ICIP) 2015, Quebec city.
- *Numerical inversion of circular arc Radon transform*
  **Syed T. A.**, Krishnan V. P. and Sivaswamy J.
  *(Under review)*.

Thank You