Tomographic Image Reconstruction in Noisy and Limited Data Settings.

Thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science in Electronics And Communication Engineering

by

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CERTIFICATE

It is certified that the work contained in this thesis, titled *"Tomographic Image Reconstruction in Noisy and Limited Data Settings."* by *Syed Tabish Abbas*, has been carried out under my supervision and is not submitted elsewhere for a degree.

Date

Advisor: Prof. Jayanthi Sivaswamy

To Ami for believing in me

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ABSTRACT

Reconstruction of images from projections lays the foundations for computed tomography (CT). Tomographic image reconstruction, due to its numerous real world applications, from medical scanners in radiology and nuclear medicine to industrial scanning and seismic equipment, is an extensively studied problem. The study of reconstructing function from its projections/line integrals, is around a century old. The classical tomographic reconstruction problem was originally solved 1917 by J. Radon, proposing and inversion method now known as filtered backprojection (FBP). It was later shown that infinitely many projections are required to reconstruct an image perfectly. It is understood that incomplete data would leads to artifacts in the reconstructed images. In addition to the artifact problem, arising due to limited data availability, the reconstructed images are known to be corrupted by noise. We study these two problems of noisy and incomplete data in the following two setups.

Nuclear imaging modalities like Positron emission tomography (PET) are characterized by a low SNR value due to the underlying signal generation mechanism. Given the significant role images play in current-day diagnostics, obtaining noise-free PET images is of great interest. With its higher packing density and larger and symmetrical neighbourhood, the hexagonal lattice offers a natural robustness to degradation in signal. Based on this observation, we propose an alternate solution to denoising, namely by changing the sampling lattice. We use filtered back projection for reconstruction, followed by a sparse dictionary based denoising and compare noise-free reconstruction on the Square and Hexagonal lattices. Experiments with PET phantoms (NEMA, Hoffman) and the Shepp-Logan phantom show that the improvement in denoising, post reconstruction, is not only at the qualitative but also quantitative level. The improvement in PSNR in the hexagonal lattice is on an average between 2 to 10 dB. These results establish the potential of the hexagonal lattice for reconstruction from noisy data, in general.

In the limited data scenario we consider the Circular arc Radon Transform (CAR). Circular arc Radon transforms associate to a function, its integrals along arcs of circles. The

transforms involve the integrals of a function f on the plane along a family of circular arcs. These transforms arise naturally in the study of several medical imaging modalities including thermoacoustic and photoacoustic tomography, ultrasound, intravascular, radar and sonar imaging. The inversion of such transforms is of natural interest. Unlike the full circle counterpart – the circular Radon transform – which has attracted significant attention in recent years, the circular arc Radon transforms are scarcely studied objects. We present an efficient algorithm that gives a numerical inversion of such transforms for the cases in which the support of the function lies entirely inside or outside the acquisition circle. The numerical algorithm is non-iterative and is very efficient as the entire scheme, once processed, can be stored and used repeatedly for reconstruction of images.

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Chapter 1

Introduction

In solving a problem of this sort, the grand thing is to be able to reason backwards. That is a very useful accomplishment, and a very easy one, but people do not practise it much. In the every-day affairs of life it is more useful to reason forwards, and so the other comes to be neglected. There are fifty who can reason synthetically for one who can reason analytically...Let me see if I can make it clearer. Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backwards, or analytically.

-Sherlock Holmes, A Study in Scarlet

An *inverse problem* is a problem of deduction of parameters of a system from the measurements we can make about the system. As opposed to an *inverse problem*, a *direct problem* is one where we predict the output given the model/parameters of the system. For example, determination of radiation pattern of a given antenna is a direct problem, while finding the antenna configuration given the radiation pattern is an inverse problem. All inverse problems occur in such dual inverse-direct pairs. Inverse problems are typically *ill-posed*, a property which is opposite to *well-posed* defined by Jacques Hadamard[1]. A problem is said to be *well-posed* in the Hadamard sense when its solution i) *is unique and exists for arbitrary data*, ii) and depends *continuously* on the data. Inverse problems arise naturally in various fields of science. Tomography or Image reconstruction from projections, is an instance of an inverse problem which arises in Medical imaging domain. In this case the *direct problem* is the measurement of projections of a function along a set of curves. Mathematically, this amounts to calculation of line-integral of the function along the curves. The desired function depends on the particular application. For example, in Computed Tomography(CT), the function of interest is the density of tissues. Similarly, in case of Positron emission Tomography(PET) the function of interest is the metabolic activity of the tissues. While the measurement of such projection require sophisticated machinery, the principle of measurement is straight forward.

Inverse tomographic problem involves reconstruction of the original density from the projection measurements. While mathematically, the direct problem is simple, the *inverse problem* however, is not as trivial. The inverse problem involves reconstruction of image from the projection data. While discussing image reconstruction from projections, one generally considers the problem of recovering some density function from measurements taken over straight lines. This problem was first discussed by Johann Radon in 1917[2] in which he discussed the recovery of a function from its line integrals. The paper lays the foundations for CT, PET, and other line integral based imaging techniques.

Multiple tomographic reconstruction methods were however developed only in 1970s by Cormak, Hounsefield etc. These methods include both iterative as well as analytic methods. Some of the most important analytical inversion methods include filtered backprojection (FBP) method developed originally by Radon[2] and Fourier series(FS) based method developed by Cormak [3], [4]. The problem of recovery of a function from its line integrals of means along different curves arise in various different scenarios in imaging. Such imaging techniques like CT, PET etc may be modelled as a Radon Transform formally defined in the next section.

1.1 Radon Transfrom

Let $f(\mathbf{x})$ be a function on \mathbb{R}^n and let \mathbb{P}^n denote the space of all hyperplanes in \mathbb{R}^n . Then *Radon transform*, $\Re f$ of $f(\mathbf{x})$ is defined as the function on the space of hyperplanes \mathbb{P}^n given by.

$$\Re f(\xi) = \int_{\xi} f(\mathbf{x}) dl \tag{1.1}$$

where, dl is a measure on the hyperplane ξ . The transform thus maps a function $f(\mathbf{x}) \in \mathbb{R}^n$ to its surface integral values over hyperplane $\xi \in \mathbb{P}^n$.

A Dual of the above transform, which is an adjoint of the forward transform, maps a function $g(\xi)$ on \mathbb{P}^n to a function \mathfrak{R}^*g on \mathbb{R}^n . The adjoint transform is given by

$$\mathfrak{R}^* g(\mathbf{x}) = \int_{x \in \xi} f(\xi) d\mu \tag{1.2}$$



FIGURE 1.1: Forward Radon Transform: The value of Radon transform at (p, ω) is obtained by integrating the function over the hyperplane ξ in direction ω at a distance of p from the origin.

where, $d\mu$ is a measure on the plane $\xi \in \mathbb{P}^n$ s.t $\mathbf{x} \in \xi$. The adjoint maps to each point \mathbf{x} the integral of all planes ξ which pass through the point.



FIGURE 1.2: Adjoint Transform : The value of adjoint transform at \mathbf{x} is computed by integrating the Radon transform of the function over all the hyperplane ξ such that $\mathbf{x} \in \xi$.

In an imaging scenario we typically measure the data in the form of line integral along different curves such as line in case of CT, circular arcs in case of ultrasound and seismic imaging, spheres in case of Thermo-acoustic/Photo-acoustic imaging etc. Given the ubiquity of the problem in various fields, numerous algorithms, both iterative as well as analytical, have been proposed for inverting the Radon transform. In the following sections we discuss two such analytical methods.

1.2 Backprojection Algorithm

Let f(x, y) be an arbitrary function in \mathbb{R}^2 . Then the *Radon transform*, $\Re f$ of f(x, y) is defined as follows.

$$\Re f(\phi, \rho) = g(\phi, \rho) = \int_{l(\phi, \rho)} f(x, y) dl$$
(1.3)

where, $l(\phi, \rho)$ is the line normal to unit vector $(\cos \phi, \sin \phi)$ at a distance of ρ from the origin, and dl is the measure along the line. Note that the line integral (1.3) represents the projection of the function f(x, y) along the line $l(\phi, \rho)$.

The above equation may also be rewritten as follows

$$g(\phi,\rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - p)dxdy$$
(1.4)

where $\delta(.)$ is the dirac delta function.

The function $g(\phi, \rho)$ is known as *The Radon Transform* of the function f(x, y). We use the terms *Radon Transform*, *Radon Data* as well as *projection data* interchangeably. Given the projection data $g(\phi, \rho)$, the back projection operator is defined as follows.

$$\hat{f}(x,y) = \int_{0}^{\pi} g(\phi, x\cos\phi + y\sin\phi)d\phi$$
(1.5)

The Backprojection operation gives an approximate inverse of the transform. The equation (1.5) essentially states that, value of the original function f(x, y) may be approximated by summing up (integrating) the values of all projection lines which pass through the point (x, y).

1.3 Fourier Series Based Inversion

An alternate approach to finding an approximation to the original function f(x, y) is based on expanding the function f(x, y) into a series. The method was discovered by Cormack in his famous work [3], [4] where he used Fourier series expansion of the function. The work eventually led to Cormak along with Hounsfield winning the 1979 Nobel Prize in Physiology or Medicine. We first note that that on conversion to the polar coordinate system, function f is periodic in the angular variable with a period 2π . Hence, we can expand the function in a Fourier series.

If $f(r, \theta)$ is the function f(x, y) in the polar coordinate system, then we have

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}$$
(1.6)

where, the Fourier coefficients $f_n(r)$ are given by

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) e^{-in\theta} d\theta$$

Similarly, the Radon transform $g(\rho, \phi)$ can also be expanded into a Fourier Series of same form as given below.

$$g(\rho,\phi) = \sum_{n=-\infty}^{\infty} g_n(\rho) e^{in\phi},$$
(1.7)

$$g_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} g(\rho, \phi) e^{-in\phi} d\phi.$$

Taking the Radon transform of the Fourier series expansiton of f (given in equation (1.6)) we have

$$g(\rho,\phi) = \sum_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{-\infty}^{\infty} f_n(r) e^{in\theta} \delta(\rho - r\cos(\theta - \phi)) r dr d\theta$$

where, $\delta(\cdot)$ is the dirac delta function. let $\beta = \theta - \phi$ then,

$$g(\rho,\phi) = \sum_{\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} rf_n(r)dr \int_{0}^{2\pi} e^{in\beta} \delta(\rho - r\cos(\theta - \phi))d\beta$$

comparing with equation (1.7) we have,

$$g_n(\rho) = \int_{-\infty}^{\infty} r f_n(r) dr \int_{0}^{2\pi} e^{in\beta} \delta(\rho - r\cos(\theta - \phi)) d\beta$$
(1.8)

Simplifying equation (1.8) we obtain a relation between the Fourier coefficients $(f_n(r))$ of the function and its transform $(g_n(\rho))$ in terms of Tchebycheff polynomials of first kind.

$$g_n(\rho) = 2 \int_{\rho}^{2\pi} f_n(r) T_n\left(\frac{\rho}{r}\right) \left(1 - \frac{\rho^2}{r^2}\right)^{-\frac{1}{2}} dr, \quad \rho \ge 0$$
(1.9)

The details of the above derivation may be found in [3], [4].

Equation (1.9) may be used to invert the function in the Fourier domain. The original

function $f(r, \theta)$ in the spatial domain can be recovered by computing the inverse Fourier series of $\{f_n(r)\}$.

1.4 Problem Statement

In this thesis we study the two analytical reconstruction methods, namely back projection based and Fourier series based method. As discussed in section 1.1, the projection data is acquired in the radon data space (\mathbb{P}^n). Conventionally, during the back projection process, image is reconstructed onto a discrete square grid. However, an alternate lattice (hexagonal) may be used instead of the conventional square lattice for reconstructing the image. In the first part of the thesis we explore this scantly studied area of tomographic image reconstructed image. We explore this question in the context of PET image reconstruction, which are known for their noisy character. We show that image quality is significantly improved by switching to the hexagonal lattice.

In the second part of the thesis we study image reconstruction in limited data scenario. It is known that image reconstruction from limited data leads to artifacts in the reconstructed image. We study the *circular arc Radon transform*, a limited data case of the full circular Radon transform, and propose a Fourier series based reconstruction algorithm for reconstructing the image form projections along arcs of fixed length. Due to the availability of only limited data, there are severe artifacts in the reconstructed images. We study the effect of various parameters on these artifacts and propose a method to suppress the artifacts.

1.5 Contributions

Both the back projection and Fourier series methods for inversion of Radon transform have been extensively studied in different contexts and settings. By early 1990's the problem of analytical image reconstruction was considered to be a well understood field, with FBP being a standard analytical algorithm. In this thesis we have studied two different analytical reconstruction techniques namely backprojection based and Fourier series based. In the first part of the thesis we consider the backprojection based algorithms for linear and arc Radon transform, and in the second part we consider Fourier series based solution of the the arc Radon transform and propose a artifact reduction algorithm for the same. In these two strategies for analytical image reconstruction methods, we claim the following contributions.

i) Image reconstruction and denoising for Hexagonal lattices. We consider linear Radon transform and propose reconstruction and denoising on hexagonal lattices based on Filtered Back Projection for reduction of noise and improvement of quality of Positron Emission Tomography(PET) images.

- ii) *Circular arc Radon transform.* We propose a new Circular Arc Radon (CAR) transform, a generalization of full circular Radon transform. We also propose a back projection based approximate inversion method for the same.
- iii) Fourier Series based inversion of CAR Transform. We numerically invert CAR transform using the Fourier series method, and discuss different parameters affecting the quality of reconstructed image.
- iv) Arifact suppression algorithm for Fourier series based inversion. Due to the partial data acquired in CAR transform, a lot of artifacts are observed in the reconstructed image. We propose an algorithm for suppression of artifacts which arise due in the reconstruction process.

Chapter 2

Image Reconstruction on Hexagonal Lattices

The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.

– MARCEL PROUST, La Prisonnière

Tomographic image reconstruction is a classical problem in image processing. It continues to be an active area of research. Due to the ill-posed nature of the problem, the reconstruction is generally very noisy. Image denoising is critical in the medical domain where images are typically obtained via a reconstruction process. A variety of techniques for denoising have been proposed recently based on Non-Local means [5] wavelets [6], curvelets [7], total variation [8] and sparse representation/Dictionary learning [9]. Depending on the nature of the modality and acquisition methodology, the reconstructed images are corrupted with noise. For instance, the need to minimize exposure (or dosage) levels of a subject to ionizing radiations such as X-ray employed in computed tomography (CT), invariably incurs a low SNR. The quality of the reconstructed image plays a key role in its usefulness as a basis for medical diagnostics. Better image quality naturally facilitates more accurate diagnosis.

In nuclear imaging, the problem is especially acute since the acquired signal is based on low photon counts that result from a radioactive decay process. Due to the randomness involved in the decay process, the noise problem cannot be alleviated by merely improving the sensor mechanism such as employing photo multipliers. Hence, this is handled at the signal processing level. Recently, the low SNR problem has been tackled with compressive sensing (CS) based approaches. CS solutions incorporating sparse constraints have been used both during and post reconstruction. Examples of the former are low



FIGURE 2.1: *Proposed Pipeline*: We propose a change of underlying reconstruction lattice from reconstruction from square(conventionally used) to Hexagonal.

dose CT [10] and PET [11] reconstruction with undersampling. Examples of the latter are the deblurring solutions proposed in [12],[13].

We argue that there is an alternative avenue for solving the noise problem, namely, by employing the hexagonal sampling lattice and demonstrate a dictionary based approach to denoising of PET images. Hexagonal lattices offer consistent connectivity and superior angular resolution motivating their study for several applications such as edge detection, morphological processing, etc., [14], [15]. The utility of this lattice in reconstruction has not been reported in literature barring a method for CT reconstruction which reports improved efficiency and memory management with hexagonal lattice [16]. Since optical cameras acquire images sampled on a square grid, resampling is required to consider the hexagonal grid-based solutions, thus limiting their practical application. However, this is not the case with PET (or CT) images, as the signal is acquired as a sinogram first thus permitting the choice of the hexagonal lattice more readily for reconstructing and denoising the final image. We choose a sparse dictionary based approach for denoising since it has been shown to perform well on natural images [9] as well as MR and fluorescence microscopy images [13]. Our approach does not incorporate the noise model in the dictionary learning step in order to clearly assess the role the change of lattice in PET image denoising using the simplest possible pipeline: reconstruction onto a hexagonal lattice using filtered back projection (FBP) followed by sparse dictionary-based denoising. We present results of assessing the denoising performance across lattices using 3 phantoms, one of which is analytical and the other two being standard phantoms used for PET reconstruction studies: the Shepp-Logan (analytical) NEMA and Hoffman. In the next section we introduce various terminology and notation used in the context of hexagonal lattices.

2.1 Hexagonal lattice definitions

Sampling lattice

A 2D image is modeled as a continuous function in \mathbb{R}^2 . Consider a continuous function $f_c(x_1, x_2)$ defined in \mathbb{R}^2 . To represent the function on a digital computer, domain of f_c on \mathbb{R}^2 must be divided to discretize the function and quantized. This division of space while discretising a function is referred to as a *tiling*. A straight forward way of sampling a 2D function is to use a rectangular (square) grid such that the sampled function becomes, $f(n_1, n_2) = f(n_1T_1, n_2T_2) = f(\mathbf{V}_s\mathbf{n})$. where $n_1, n_2 \in \mathbb{Z}$. In this square sampling case, the matrix \mathbf{V}_s is the standard Euclidean basis (2.1). However, any valid basis \mathbf{V} , of Euclidean plane may be used to sample the \mathbb{R}^2 plane. Any sampling point is then an integer linear combination of the basis vectors (column vectors of \mathbf{V}). Horizontally aligned hexagonal lattices may be generated using \mathbf{V}_h shown in equation (2.2).

$$\mathbf{V}_s = (e_1, e_2) = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$
(2.1)

$$\mathbf{V}_{h} = (h_{1}, h_{2}) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$
(2.2)

Based on the sampling basis **V** used, we can define a *neighborhood* \mathcal{N} and *packing density* of the sampling lattice.

Sampling density of a lattice is defined as $|det\mathbf{V}|$, where \mathbf{V} is the sampling basis used. The hexagonal lattice has a higher density that a square lattice as shown below:

$$|det \mathbf{V}_h| = \frac{\sqrt{3}}{2} \le |det \mathbf{V}_s| = 1 \tag{2.3}$$

Neighborhood of a point in a lattice **V** may be defined using the basis vectors pointing in the direction of the neighboring lattice points[17]. For a square lattice two commonly use neighborhoods are 4-neighborhood(\mathcal{N}_s^4) and 8-neighborhood(\mathcal{N}_s^8). These may be formalized by defining them as follows.

$$\mathcal{N}_s^4 = \{e_1, e_2\}$$

 $\mathcal{N}_s^8 = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$

Set \mathcal{N}_s^4 has 2 elements which represent 4 neighbors at location $\mathbf{n} \pm e_1$, and $\mathbf{n} \pm e_2$ for each lattice site $\mathbf{n} \in \mathbb{Z}^2$. Similarly, set \mathcal{N}_s^8 represents 8 neighbors of any lattice site. Similarly, for a hexagonal lattice the neighborhood is defined as the following set of



FIGURE 2.2: Hexagonal image addressing shown for an order 2 image

$$\mathcal{N}_{h}^{6} = \{h_1, h_2, h_1 - h_2\}$$

It should be noted that $|h_2 - h_1| = |h_1| = |h_2|$ and hence all the neighboring sites in a hexagonal 6 neighborhood are equidistant. In contrast, in a square lattice $(e_1 + e_2 = |e_1 - e_2|) = \sqrt{2}(|e_1| = |e_2|)$. This shows that while each of 4-neighborhood is symmetrical, 8-neighborhood (\mathcal{N}_s^8) is asymmetrical. This symmetry in distance of neighbors is one of the superior property of hexagonal lattices.

Addressing in hexagonal lattice

It can be seen clearly from \mathbf{V}_h that the Cartesian coordinates of the sampling points in the hexagonal grid are $(x_i, y_j) \notin \mathbb{Z}^2$ as is the case for the square lattice. In the former case unlike the latter, these have irrational values. In addition to being less intuitive, using irrational numbers to index lattice is not practical because of various issues such as limited floating point precision on a digital computer, difficult computations involved. Thus, sampling an image onto a hexagonal grid requires a new addressing/indexing scheme. Various addressing schemes have been proposed in literature based on tricoordinates systems [18], [19]. We choose to follow the single-index addressing scheme proposed in [20] and [21] as it has been shown to be efficient. Both these methods are based on a spiral addressing scheme. The method facilitates the representation of 2D image using a 1D array. The proposed indexing method makes use of hierarchical aggregates to exploit the size of \mathcal{N}_{h}^{6} neighborhood and uses a base-7 addressing scheme as shown in figure 2.2. This scheme gives rise to images with 7^l pixels where l is referred to as the order or level. Owing to the 1D nature of the resultant data structure, it is easier to vectorize the image and many operations are simplified. Based on the addressing scheme, we now define a patch in an image defined on square and hexagonal lattices. A patch in an image defined on a square lattice at a location $l \equiv (i, j)$ and size $n \times n$ may be expressed as follows,



FIGURE 2.3: Neighborhood of location 42_7 , shown in relation with the origin 0_7



FIGURE 2.4: Vectorizing a patch of order 2

$${}^{s}\mathcal{P}_{l}^{n} = \left\{ (i - \lfloor \frac{n-1}{2} \rfloor, j - \lfloor \frac{n-1}{2} \rfloor), ..., (i,j), ..., (i - \lfloor \frac{n}{2} \rceil, j - \lfloor \frac{n}{2} \rfloor) \right\}$$

In a similar way, we can define a patch on a hexagonal grid. However, due to the use of linear indexing, the definition of an image patch in hexagonal case is much simpler. We define a hexagonal *patch* of order n centred at location l_7 as the set of pixels given by:

$${}^{h}\mathcal{P}_{l}^{n} = \{l_{7}, l_{7} + 1_{7}, l_{7} + 2_{7}, ... l_{7} + (7^{n})_{7}\}$$

For example a patch of order 1 at location 42_7 is given by:

$${}^{h}\mathcal{P}^{1}_{42} = \{42, 5, 4, 36, 43, 40, 41\}$$

Note that a patch of order 1 will have $7^1 = 7$ elements, of order 2 will have $7^2 = 49$ elements, and so on. It should be noted that a hexagonal patch of order 2 is equivalent to a square patch of size 7×7 in terms of the number of pixels.

2.1.1 Why Hexagonal lattice?

Using Hexagonal lattice has some clear disadvantage of non-alignment with Cartesian coordinates, which leads to difficulties in doing calculus on the lattice. Further, an extension to higher dimensional images is not straightforward. However, it can be quite

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beneficial in some cases (ex. medical image reconstruction) where the data is not acquired onto square grid, as in the case of natural images. It was pointed out earlier in section 2.1 that, unlike square lattice, each of the (six) nearest neighbors (\mathcal{N}_h^6) in a hexagonal lattice are equidistant. This property of equidistant neighbors implies that curves may be represented in a better fashion on a hexagonal lattice [20].

Additionally, the packing density of hexagonal lattice is $\sqrt{3}/2$ times greater (see Equation (2.3)) than the square lattice. In fact of all possible tilings of 2D plane, for a given perimeter of tile, hexagon has the maximum area which is the famous 'honeycomb conjecture' [22]. These two properties indicate that there is a greater redundancy of representation in the hexagonal image. This, can lead to an increased robustness to degradation in general. Also, various structures in images, which are bounded by some edges, are better represented on hexagonal grid. Redundancy of a representation dictates the degree of robustness of a signal to degradation. Recovery of a signal from a more redundant representation is relatively easier. While a redundant signal is more compressible, it is also easier to predict degraded/unknown values of the signal. In this sense, the ideas of redundancy, compressibility and ability to recover a signal from its degraded version are very closely linked. It may be noted that if a class of signals (e.g. natural images) is 'sparse' in a given basis (e.g wavelet basis in case of natural images), the basis may be used for denoising of the signal (e.g. wavelets, curvelets etc). Many de-noising schemes exploit this relation between redundancy, sparsity and ability of signal recovery. For example, denoising schemes based on wavelet thresholding for natural images, or curvelet/ridgelet based denoising are based on the ability of these bases to sparsify images. A more elaborate discussion on this relation may be found in [23].

2.2 PET reconstruction and Dictionary based denoising

PET is a nuclear imaging modality used to study functional activities of living tissues such as glucose metabolism etc. It is an emission modality which is invasive in the sense that it involves injection of positron emitting radio-isotopes into the patient. The injected radio-isotope decays during its metabolism inside the tissues, emitting positrons. These positrons annihilate after colliding with their anti-particle, electron. The γ -photons emitted as result travel in radially opposite directions which are detected outside the body by the PET scanner.

The emitted photons (each of energy 511 KeV) however interact with body tissue in their ideal straight line path. Compton scattering is one of the most prominent interaction that the photons undergo. These interactions lead to photons losing some energy as well as changing their direction. This scattering effect leads to 'attenuation' of photons, which is the major cause of degradation in PET images.

The attenuation correction factors for PET are estimated using external photon sources



FIGURE 2.5: NEMA phantom before and after attenuation correction

with the body present and comparing it with ones without any body present in the scanner. In addition to the attenuation correction, sinogram also needs to be corrected for noise arising from scatter(due to interaction of photons with body) and random events such as other positrons annihilating at the same time. In this work, we do not attempt to correct this error or the errors which arise due to non-uniformity of detectors of PET scanner. The proposed pipeline for denoising has two stages: In stage 1, the PET image is reconstructed onto a desired lattice from the given sinogram and in stage 2 the reconstructed image is denoised using the KSVD algorithm. Details of these stages are explained next.

2.2.1 Filtered Back Projection

Filtered back projection (FBP) is an analytical reconstruction method, which essentially is an algorithm for inverting the sinogram which is the Radon transform of the desired image or a function f(x, y). FBP is a basic algorithm for image reconstruction. Since the sinogram data is obtained by forward projection of a function, intuitively, the inversion of this process can be done by backprojecting the projection data. However, the cumulative summation during backprojecting can lead to boosting of the low frequency content. This results in the output of simple backprojection to be blurred in practice. To correct the blur, the output is filtered with a ramp shaped filter to suppress the low frequencies, giving rise to the name *filtered* back projection. Figure 2.6 illustrates graphically the filtered back-projection algorithm. The details of the mathematics of FBP algorithm may be seen from Appendix B. FBP assumes the data to be noise free and hence will lead to noisy reconstructions given noisy sinograms. Hence, in practice, iterative, statistical reconstruction methods like [24], [25], [26] etc are employed to achieve better SNR. However, given our focus in the present work on the role of lattice, FBP serves as worst-case baseline algorithm yielding a noisy reconstruction.

2.2.2 Denoising

The denoising is based on the KSVD algorithm [27]. This algorithm is based on a Dictionary learned over patches drawn from the given noisy image. The main steps



FIGURE 2.6: Graphical representation of Filtered Back Projection

in the algorithm are: Dictionary learning, sparse coding, reconstruction of the denoised image using sparse code and the learnt dictionary. The denoising is based on the sparsity of the image which means fewer atoms capture the 'clean' signal which is not the case with the noise component.

Learning over-complete sparse dictionary

Various methods [28] [29] [30] etc have been proposed for learning sparse dictionaries. We have used the 'online approach'[30] for learn a dictionary which we briefly review next. Randomly sampled patches ${}^{h}\mathcal{P}_{l}^{n}$, ${}^{s}\mathcal{P}_{l}^{n}$ from images are vectorized to generate training data for training the dictionary. The algorithm in [31] is used for learning the dictionaries and solve the optimization problem below.

$$\mathcal{C} = \{ \mathbf{D} \in \mathbb{R}^{m \times k} s.t \forall j = 1, \dots k, d_j^T d_j \le 1 \}$$
(2.4)

$$\min_{D \in \mathcal{C}, \alpha \in \mathbb{R}^{k \times n}} \frac{1}{2} \| \mathbf{X} - \mathbf{D}\alpha \|_F^2 + \lambda \| \alpha \|_{1,1}$$
(2.5)

where, $\mathbf{X} = (x_{s(h)1}, x_{s(h)2}, \dots x_{s(h)n}), \alpha$ is the sparse codes of the vectors, $\|\cdot\|_F$ is the Frobenius norm and $\|\cdot\|_{1,1}$ is the l_1 norm. Two dictionaries \mathbf{D}_h (hexagonal) and \mathbf{D}_s (square) are learnt using (2.5). The method described is fast and optimized for large training set (which is the case for densely sampled image patches). We have used a batch size of 400 and $\lambda = 0.6$ while training the dictionary. An example of individual atoms learned in the hexagonal case is shown in figure ??. Sparse coding is done using Cholesky factorization-based orthogonal matching pursuit of the test signals. The algorithm efficiently computes in parallel, the sparse codes α which optimize one of the following equations

$$\min_{\alpha} \| \alpha \|_0 \ s.t \| x - D\alpha \|_2^2 \le \epsilon \tag{2.6}$$

$$\min_{\alpha} \| x - D\alpha \|_{2}^{2} s.t. \| \alpha \|_{0} \le L$$
(2.7)

$$\min_{\alpha} \frac{1}{2} \parallel x - D\alpha \parallel_2^2 + \lambda \parallel \alpha \parallel_0$$
(2.8)

Learning an over-complete dictionary is done from patches of the noisy reconstructed image. As pointed out in [27], the sparse dictionary so obtained is tuned to the particular example without having to assume universality of sparsity. Training data for learning is obtained in the form of vectorized patches of the images in respective grids. In the case of square lattice, the patch ${}^{s}\mathcal{P}_{l}^{n}$ is vectorized by row-major ordering to obtain a vector $\mathbf{x}_{s} \in \mathbb{R}^{n^{2}}$. For the hexagonal case, we use the spiral indexing method (see section 2.1). Each patch ${}^{h}\mathcal{P}_{l}^{n}$ is converted into a vector $\mathbf{x}_{h} \in \mathbb{R}^{7^{n}}$ (see figure 2.4). Fair comparison of lattices was ensured by fixing n = 2 in our experiments, to get a 49-dimensional vector for both lattices.

Vectors $\{\mathbf{x}_h\}_{i=1}^N$ and $\{\mathbf{x}_s\}_{i=1}^N$ obtained by randomly sampling the patches from the noisy image are used to learn two dictionaries \mathbf{D}_h and \mathbf{D}_s . These learned dictionaries are then used for denoising. A (sparse code) weighted combination of dictionary atoms are averaged to obtain the final denoised image.

2.3 Experiments and results

The method was assessed with 3 standard phantom images: i) Shepp-Logan phantom, which is analytically derived and routinely used to evaluate reconstruction algorithms ii) the NEMA and Hoffman brain phantoms which are specifically used to test PET reconstruction. In order to display the reconstructed results on hexagonal lattice, the pixels were visualized with square hyper-pixels using the code provided in [14]. The Shepp-Logan phantom, unlike the the other two, permits a controlled study of denoising. In our experiments, first, the phantom image I (generated using Matlab) was degraded with additive Gaussian noise to model the noisy source I_n . Next, the sinogram, constructed by computing the Radon transform of I_n , was used to reconstruct noisy images I_r onto square/hexagonal lattices. Finally, I_r was denoised in the native lattices. The original image I, its noisy reconstructions I_r and the denoised results are shown in Figure 2.9 for standard deviation 0.08. From these results, we see that the central small, white, circle has better definition and shape fidelity on the hexagonal compared to the square lattice in Nema phantom. (Figure 2.8) Further figure 2.7 illustrates the shape fidelity in Hoffman Phantom. The denoised image on the hexagonal grid is smoother as well.



FIGURE 2.7: Robustness of shape recovery in Hexagonal lattice (Row 1) compared to Square lattice (Row 2) for Hoffman phantom



FIGURE 2.8: Robustness of shape recovery in Hexagonal lattice (Row 1) compared to Square lattice (Row 2) for NEMA phantom



FIGURE 2.9: Comparison of denoising results for the Shepp-Logan phantom. Clockwise from top left: original, noisy image, denoised results on hexagonal and square grids

The denoising was assessed quantitatively by varying the noise and computing the PSNR with I as the 'clean' original. The experiments were repeated 5 times and the average values were recorded. Figure 2.10a shows this average PSNR as a function of noise levels. A trend analysis of the plot shows that, for high PSNR (i.e. low noise levels) a change to hexagonal lattice results in a 5 dB improvement in denoising while for high noise levels, the improvement is half as much.

Figure 2.12 shows the noisy reconstructed (I_r) and denoised results (I for the NEMA and Hoffman phantoms. A quantitative assessment of NEMA phantom was done in two ways:

a) An 'inverse' PSNR metric, which treats the denoised image as the clean signal and the noisy reconstruction as the 'noisy signal', was computed. A large magnitude of 'error' indicates good denoising. The average (over 5 repetitions) inverse PSNR for the NEMA phantom were -59.7 dB and -51.6 dB for the square and hexagonal lattices, respectively (Figure 2.10b). For the Hoffman phantom these values were -51.5 dB (square) and -41.5 dB (hexagonal). This demonstrates that the improvement in denoising with hexagonal lattice is between 8 to 10 dB.

b) The intensity profile along several scan lines in the denoised image were analysed. This was done only for NEMA phantom as it is the standard used for PET calibration. A scan line profile is shown in Figure 2.11. The line position, as indicated in the inset image, covers two objects of opposite polarity on a noisy background. Hence, the ideal profile should be flat at the location of the objects. This is the case especially for the bright object in the hexagonal lattice whereas it is not in the square lattice. The region between bright and dark objects represent the background which appears noisier in the square case both in Figure 2.12 and the profile in Figure 2.11. Thus, the hexagonal



FIGURE 2.10: PSNR Analysis for Shepp-logan and NEMA Phantom



FIGURE 2.11: Scanline comparison for NEMA on **Hexagonal (red)**, **Square (blue)** lattices. Inset image shows the scan line. The labelled pixel positions in the plots are with the origin at the centre of the image

lattice appears to be better at preserving the fidelity of the shape after reconstruction and denoising.



FIGURE 2.12: Comparison of denoising results for NEMA and Hoffman phantoms. Left to right: Noisy reconstruction, denoised images on hexagonal and square lattices.



FIGURE 2.13: Comparison of denoising results (clockwise from top left, original, denoised on square grid, denoised on hexagonal grid, noisy image)

2.4 Conclusions and Future Directions

In this chapter we discussed that an alternate solution to improving image denoising in reconstructed images is to change the underlying sampling lattice. We have done this by extending the adaptive dictionary based denoising to hexagonally sampled images. The experimental results demonstrate that using a hexagonal lattice for reconstruction and denoising of PET sinogram data improves the performance of reconstruction both qualitatively as well as quantitatively. The denoising methodology was tested on simple FBP reconstruction algorithm rather than using more sophisticated techniques to demonstrate the power and usefulness of the proposed idea.

While this present work focuses on the specific modality of PET imaging, we also note

that similar improvements are to be had even on natural images (see figure 2.13). There are various future directions that can be explored

- a) Explicit noise/degradation modelling can be incorporated into the dictionary learning scheme to improve the results.
- b) An iterative statistical technique for reconstruction from sinogram.
- c) Dictionary learning based reconstruction methods can be adapted to hexagonal lattice for the benefits demonstrated here and reported elsewhere in literature or,
- d) The mathematical properties of hexagonal lattice with regard to its ability to (sparsely) represent and recover degraded signals.

The sampling lattice has been known to play a role in digital image processing for almost three decades. A lot of investigation needs to be done to explore the role of lattice in basic image processing operations.

Chapter 3

Circular Arc Radon Transform: Definition and back-projection based inversion

We're not retreating, we're advancing in reverse.

-Skulduggery Pleasant, Playing with Fire

Circular arc Radon (CAR) transforms arise naturally in the study of several medical imaging modalities including thermoacoustic and photoacoustic tomography, ultrasound, intravascular, radar and sonar imaging. For a function f on the plane, CAR transforms involve integrals of f along a family of circular arcs. In order to motivate the study of CAR transforms, we begin with circular/spherical Radon transforms whose study has turned out to be of immense interest in the aforementioned imaging modalities. For example, Thermoacoustic/Photoacoustic tomography (TAT/PAT) is a recent method with potential applications in medical imaging. In TAT/PAT the object of interest is irradiated by a short electromagnetic (EM) pulse. The irradiated tissue absorbs some of the EM energy. Depending on the anatomical, structural, functional and metabolic characteristics, different tissues absorb different amount of EM energy depending on the concentration of various *chromophores* such as haemoglobin, melanin, lipids etc. The concentration of these chemicals indicate any physiological changes, such as oxygen saturation, vascular blood volume in the body etc [32, 33]. Cancerous cells, for example, absorb more energy than the healthy cells due to high metabolic activity. Therefore, it is diagnostically useful to know the absorption function of tissues [34] [35] [36]. This



FIGURE 3.1: Diagrammatic representation of setup for photo-acoustic measurement.

absorption of EM pulse causes an increase in the local temperature, and makes the tissues expand. This elastic expansion caused by absorption of EM pulse causes a pressure distribution in the tissue, which is roughly proportional to the absorption function. This initial pressure, in turn, leads to a pressure wave p(t, x) which propagates through the object, and is then measured by transducers located on an observation surface P surrounding the object (see figure 3.1). The goal is to use the measured data to reconstruct the initial pressure p(0, x). An accepted model [37] for pressure waves p(x, t) arising in TAT/PAT setups is

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} = c^2(x)\Delta_x p, & t \ge 0, x \in \mathbb{R}^3\\ p(x,0) = f(x), \frac{\partial p(x,0)}{\partial t} = 0. \end{cases}$$
(3.1)

Where c(x) is the speed of the acoustic wave. The initial value of pressure, f(x) is the function of interest, which represents the image. The data received at the the detectors, located on the surface P is given by

$$g(y,t) = p(y,t), \quad y \in P, t \ge 0$$

For real transducers, located along a circle P the pressure measurement may be possible only over a limited angular span, constrained by physical setup of the equipment. If the transducers are collimated to receive data along a plane, the measured data then can be modeled as a circular Radon transform with centers on the intersection of the plane with the acquisition surface P. Assuming a simplified model that the background wave speed is a constant, the measured data can be modeled as a spherical Radon transform of the initial pressure distribution p(0, x) [37] with centers on the acquisition surface P. We assume the transducers with restricted transmission/absorption span in the angular direction located throughout uniformly on an acquisition surface P. These transducers measure the pressure p(y, t), where $y \in P$ is a detector location and $t \ge 0$ is the time of the observation. Assuming that the speed of sound propagation within the medium is constant, the medium is weakly reflecting and that the pulses radiate isotropically, the data registered at the transducer is the superposition of the pulses reflected from those inhomogeneities which are equidistant from the transducer location. In the continuous case, this leads to the consideration of the circular Radon transform of a function (which models the medium) on a plane, assuming the transducer is collimated to receive only reflections from that plane. The inversion of this transform leads to the recovery of an image of the medium.

The study of circular/spherical Radon transforms has attracted the attention of several authors in recent years [38–54]. The transform involved in this set up associates for a given function, its integrals along circular arcs with constant angular span instead of integrals along full circles. Furthermore, in some imaging problems, full data in the radial direction may not be available, for instance, in the case of imaging the surrounding region of a bone. We consider these two imaging scenarios in the present and the subsequent chapters.

3.1 Circular Arc Radon Transform

Let (r, θ) denote the standard polar coordinates on the plane \mathbb{R}^2 and let $f(r, \theta)$ be a compactly supported function in \mathbb{R}^2 .

Let P(0, R) denote a circle (acquisition circle) of radius R centered at the origin O = (0, 0) and parametrized as follows:

$$P_{\phi} = (R\cos\phi, R\sin\phi) : \phi \in [0, 2\pi].$$

We consider the circular arc Radon(CAR) transform $(\mathcal{R}_{\alpha}f)(\rho,\phi)$ of function $f(r,\theta)$ along circular arcs of fixed angular span α . The details of the setup are illustrated in Figure 3.2.

Let $C(\phi, \rho)$ be a circle of radius ρ , centered at P_{ϕ} . That is,

$$C(\rho, \phi) = \{ (r, \theta) \in \mathbb{R}^2 : |x - P_{\phi}| = \rho \}$$

Let $A_{\alpha}(\rho, \phi)$ be an arc of the circle $C(\rho, \phi)$ with an angular span of 2α . This is given by

$$A_{\alpha}(\rho,\phi) = \{(r,\theta) \in \mathbb{R}^2 : |x - P_{\phi}| = \rho, \theta \in [\phi - \alpha, \phi + \alpha]\}.$$

We define the circular arc Radon transform of f over the arcs $A_{\alpha}(\rho, \phi)$ as follows:

$$g^{\alpha}(\rho,\phi) = \mathcal{R}_{\alpha}f(\rho,\phi) = \frac{1}{2\alpha\rho} \int_{A_{\alpha}(\rho,\phi)} f(r,\theta) \,\mathrm{d}s, \qquad (3.2)$$

where ds is the arc length measure on the circle $C(\rho, \phi)$ and $A_{\alpha}(\rho, \phi)$ is the arc over which the integral is computed (see Figure 3.2). The full circular Radon transform is



FIGURE 3.2: Setup for object supported inside the acquisition circle, where P_{ϕ} is the detector located at an angle ϕ on the acquisition circle.

defined by the following integral

$$g(\rho,\phi) = \mathcal{R}f(\rho,\phi) = \int_{C(\rho,\phi)} f(r,\theta) \,\mathrm{d}s, \qquad (3.3)$$

Note that the difference between the full circular Radon transform and the Circular arc Radon transform defined in (3.2) is in the limits of integration.

3.2 Discrete Circular Arc Radon Transform

The forward transform is computed by discretizing Equation 3.2. We approximate the integral by the following sum (Equation (3.2)). The discrete transform is computed for $\rho \in [0, R - \epsilon]$, i.e data comprises of integrals from acquisition circle P(0, R) to the origin O. We therefore have,

$$g^{\alpha}(i,j) = \frac{1}{2\alpha\rho_i} \sum_{(x_n, y_m) \in \mathcal{A}_{i,j}} f(x_n, y_m)$$
(3.4)


(A) Original Image

(B) Circular Arc Radon Transform

FIGURE 3.3: Sample images and their corresponding Circular arc Radon transform for $\alpha = 25^{\circ}$.

where

$$\mathcal{A}_{i,j} = \left\{ (x_n, y_m) : \sqrt{(x_n - R\cos\phi_j)^2 + (y_m - R\sin\phi_j)^2} = \rho_i^2, \phi_j - \alpha \le \arctan(\frac{y_m}{x_n}) \le \phi_j + \alpha \right\}$$

where $\rho_i = ih, i = 0, 1, ..., M - 1, \quad h = \frac{R - \epsilon_{\rho}}{M},$ and $\phi_j = jl, j = 0, 1, ..., N - 1, \quad l = \frac{\pi}{N}.$

Note that $g^{\alpha}(i, j)$ is an $N \times M$ matrix. Figure 3.3 shows two phantom images image f(x, y) and their corresponding Circular-Arc Radon(CAR) transform $g^{\alpha}(\rho, \phi)$. In figure 3.3 we have $\alpha = 25^{\circ}$ and M = N = 300.

3.3 Adjoint Transform.

We know that Radon transform maps a function from the Euclidean plane to a space of curves. In the case of linear transform the mapping (in 2D case) is from 2D euclidean plane to the set of lines. Similarly in the case of circular Radon transform the mapping is from the euclidean plane to the space of circles. In the Circular arc Radon transform we map the function $f(r, \theta)$ for euclidean plane to the set of arcs centred on a circle P, with a fixed angular span of 2α . We define a dual transform, which is a mapping from the set of arcs to the euclidean plane. The dual transform may be defined as given in equation (3.5) below.

$$f(x,y) = \int_{0}^{2\pi} g(\rho, \sqrt{(x - \cos \phi)^2 + (y - \sin \phi)^2}) d\phi$$
(3.5)

Equation (3.5) is analogous to the back-projection algorithm of the usual linear Radon transform. However, unlike the linear transform, values are back-projected along circles instead of lines. The back-projection algorithm in the case of linear Radon transform states that value of a function at a point (x, y) is the sum(integral) of Radon transform

values along all lines passing through the point. The details of the linear backprojection were discussed in Chapter 2 Equation (3.5) says that the value of the function at point (x, y) is the sum(integral) of all the arc-Radon values along circles passing through the point (x, y). We discretize Equation (3.5) and approximated by the integral by following sum.

$$f[n,m] = \sum_{i=0}^{N} g\left[\left(\frac{2\pi i}{N}\right), \sqrt{\left(n - \cos\left(\frac{2\pi i}{N}\right)\right)^2 + \left(m - \sin\left(\frac{2\pi i}{N}\right)\right)^2}\right]$$
(3.6)

A detailed implementation detail of the adjoint computation is given in Algorithm 1 below.

```
Data: Radon Transform matrix, g^{\alpha}(\rho, \phi) of size M \times N

Result: f(x, y)

f(r, \theta) = [0]_{M \times N}

for i < M do

\begin{vmatrix} \mathbf{for} \ j < N \ \mathbf{do} \\ var \ sum := 0 \\ \mathbf{for} \ k < N \ \mathbf{do} \\ | \ (p, q) := \left(\frac{2\pi k}{N}, \sqrt{\left(n - \cos\left(\frac{2\pi i}{N}\right)\right)^2 + \left(m - \sin\left(\frac{2\pi i}{N}\right)\right)^2}\right) \\ sum = sum + g(p, q) \\ end \\ f[i, j] = sum \\ end

f(x, y) = Polar2cartesian(f(r, \theta))

Algorithm 1: Adjoint Inversion
```

3.4 Results And Discussion

We tested the algorithm on analytical phantoms shown in figure 3.4.

Figure 3.6 and 3.5 show a few sample reconstructions for arc-Radon transform with different angular span α .

The following observations may be made.

• Un-reconstructed Edges. We observe that as the opening angle of the transform increases, more edges are reconstructed. For an edge in the image to be visible, roughly speaking, there should be an element from the data set, that is a circular



FIGURE 3.4: Phantoms used during experiments



FIGURE 3.5: Example image reconstructions using Adjoint method of Algorithm 1 for phantom 8 shown in figure 3.4



FIGURE 3.6: Example image reconstructions using Adjoint method of Algorithm 1 for phantom 7 shown in figure 3.4



FIGURE 3.7: Example image reconstructions using Adjoint method of Algorithm 1 for phantom 8 shown in figure 3.4 with high-pass filtering

arc, tangential to the edge. A formal justification of this statement follows using the tools of Fourier integral operator theory and microlocal analysis [55–57]. All other edges get blurred out in the reconstructed image. As the value of α increases, more edges become visible. As the angle α increases, more edges are tangential to the circular arc of integration, hence number of visible edges increases with α [58].

• The output image has a blurred/hazy appearance. The reconstructed images have a blurry appearance. This hazy appearance is typical to the back-projection algorithm. Taking hint for the Filtered Back-projection algorithm for linear Radon Transform, discussed in 2, we observer that the appearance may be improved to some extent by a post-processing step involving high-pass filtering the images. The high-pass filtered images are shown in figure 3.7.

Further, from equation (3.5) we observe that reconstruction algorithm essentially projects the acquired data along full circles. This is similar to the linear case, where data is projected back along whole line instead of line segments. In the next section we describe a modified algorithm to reduce the blurring of edges by modifying the back-projection process.

• Streaking Artifacts. From figure 3.7 it may be observed that while high pass filtering does enhance the real edges in the images, the arifacts in the image still persist. The presence of these streaking artifacts is due to the limited data used in reconstruction process. We further discuss artifacts in limited data tomography, in chapter 5. It may be noted that in the present setup we truncate the data in both angular as well as radial direction. In the radial direction data is truncated at a distance ρ equal to the radius of acquisition circle R. In the angular direction, the truncation is a function of angle α . Hence, we observe that the the severity of the artifact problem reduces as the α increases.

In order to improve the quality of reconstruction both in terms of sharpness as well as reduced artifacts we use a Fourier Series based inversion method. The details of the method are discussed in the subsequent chapters.



FIGURE 3.8: Example image reconstructions using Adjoint method of Algorithm 1 for phantom 8 shown in figure 3.4 with high-pass filtering

Chapter 4

Circular arc Radon Transform: Fourier series based inversion

"How dangerous it always is to reason from insufficient data."

-SHERLOCK HOLMES, The Adventure of the Speckled Band

4.1 Introduction

Circular arc Radon (CAR) transforms involve the integrals of a function f on the plane along a family of circular arcs. These transforms arise naturally in the study of several medical imaging modalities including thermoacoustic and photoacoustic tomography, ultrasound, intravascular, radar and sonar imaging. In chapter 3 we considered inversion of such a circular arc Radon transform based on back-projection operator. In the present chapter we consider an alternate solution of the problem.

The case of partial data in the radial direction for circular and elliptical Radon transforms was considered in [59, 60]. Two related works where circular arc means transform have appeared are in SAR imaging [61] and Compton scattering tomography [62]. However, their setup is different for the one considered in the current paper, in the sense that, the data is acquired along semicircular arcs of different radii in [61], while in [62] they are along circular arcs with a chord of fixed length. We consider circular arcs of a fixed angular span with a circular acquisition geometry (a typical set up in several medical imaging procedures). The setup leads to a non-standard integral transform, described in Section 4.2, which has not been considered previously. Our image reconstruction procedure is based on the derivations in [59, 60] which adopts an inversion strategy originally due to Cormack [63] that involved Fourier series techniques. Due to rotational symmetry, the n^{th} Fourier coefficient of the CAR transform data is related to the n^{th} Fourier coefficient of the unknown function by a nonstandard Volterra-type integral equation of the first kind with a weakly singular kernel. This is the point of departure from the derivations in [59, 60] where a standard Volterra-type integral equation of the first kind with a weak singularity was involved. Through the method of kernel transformation, the singularity in the integral equation can be removed [64] leading to a standard Volterra-type integral equation of the second kind. It is well known that such an integral equation has a unique solution and this can be obtained by the Picard's process of successive approximations, leading to an exact inversion formula given by a infinite series of iterated kernels; see [64].

However, in our situation, we have a Volterra integral equation of the first kind with a weakly singular kernel, in which both the upper and lower limits of the integral are functions. Such integral equations have been investigated in previous studies [65-69]. However, to the best of our knowledge, the integral equation that we have does not seem to fit into these previously established results. In the current article, we present an efficient numerical inversion of the Volterra integral equation of the first kind appearing in the inversion of the CAR transform. Our work is based on the numerical algorithm for the inversion of a Volterra integral equation recently published in [70] that used the trapezoidal product integration method [71, 72]. The inversion techniques of [70] have also been employed in the numerical inversion of a broken ray transform [73]. Unlike the situation in [70], due to the presence of the edges of the circular arcs (see Figure 4.1) and also due to the fact that the angular span of the arcs places restrictions on the edges that are visible (in the sense of microlocal analysis [56, 74]), the reconstruction algorithm introduces severe artifacts in the image. In the following chapter, we propose an artifact reduction strategy in this paper. Some recent works that deal with suppressing artifacts are [75–80].

4.2 Nonstandard Volterra integral equations

Let (r, θ) denote the standard polar coordinates on the plane and let $f(r, \theta)$ be a continuous compactly supported function on $[0, \infty) \times [0, 2\pi)$ such that $f(r, 0) = f(r, 2\pi)$ for all $r \ge 0$. Let P(0, R) denote a circle (acquisition circle) of radius R centered at the origin O = (0, 0) and parametrized as follows:

$$P(0,R) = \{ (R\cos\phi, R\sin\phi) : \phi \in [0,2\pi) \}.$$

We consider the CAR transform $(\mathcal{R}_{\alpha}f)(\rho,\phi)$ of function $f(r,\theta)$ along circular arcs of fixed angular span α . The details of the setup are illustrated in Figure 4.2.

Let $C(\rho, \phi)$ be the circle of radius ρ centered at $P_{\phi} = (R \cos \phi, R \sin \phi)$. That is,

$$C(\rho, \phi) = \{ (r, \theta) \in [0, \infty) \times [0, 2\pi) : |x - P_{\phi}| = \rho \}$$



FIGURE 4.1: Measurement Setup

Let $A_{\alpha}(\rho, \phi)$ be the arc of the circle $C(\rho, \phi)$ with an angular span of α . This is given by

$$A_{\alpha}(\rho,\phi) = \{(r,\theta) \in [0,\infty) \times [0,2\pi) : |x - P_{\phi}| = \rho, \\ \theta \in [\phi - \alpha, \phi + \alpha]\}.$$

We define the CAR transform of f over the arc $A_{\alpha}(\rho, \phi)$ as follows:

$$g^{\alpha}(\rho,\phi) = \mathcal{R}_{\alpha}f(\rho,\phi) = \int_{A_{\alpha}(\rho,\phi)} f(r,\theta) \,\mathrm{d}s, \qquad (4.1)$$

where ds is the arc length measure on the circle $C(\rho, \phi)$ and $A_{\alpha}(\rho, \phi)$ is the arc over which the integral is computed (see Figure 4.2) with $\rho \in (0, R - \epsilon), \epsilon > 0$.

Since both $f(r, \theta)$ and $g^{\alpha}(\rho, \phi)$ are 2π periodic in the angular variable, we may expand them into their respective Fourier series as follows.:

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}$$
(4.2)

$$g^{\alpha}(\rho,\phi) = \sum_{n=-\infty}^{\infty} g_n^{\alpha}(\rho) e^{in\phi}, \qquad (4.3)$$

where the coefficients $f_n(r)$, $g_n^{\alpha}(\rho)$ are given as follows:

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) e^{-in\theta} d\theta$$
$$g_n^{\alpha}(\rho) = \frac{1}{2\pi} \int_0^{2\pi} g^{\alpha}(\rho,\phi) e^{-in\phi} d\phi.$$

Based on our assumption on the f, the Fourier series of f and g^{α} will converge almost everywhere. We now use an approach similar to one followed by [59] for circular Radon transform, which is based on the classical work by Cormack [3] for the linear Radon case.

Using the Fourier series expansion of function $f(r, \theta)$ in Equation (4.1) we have

$$g^{\alpha}(\rho,\phi) = \sum_{n=-\infty}^{\infty} \int_{A_{\alpha}(\rho,\phi)} f_n(r) e^{in\theta} \mathrm{d}\theta.$$

Since the arc $A_{\alpha}(\phi, \rho)$ is symmetrical about ϕ we may rewrite the integral as follows.

$$g^{\alpha}(\rho,\phi) = \sum_{n=-\infty}^{+\infty} \int_{A^{+}_{\alpha}(\rho,\phi)} f_{n}(r) \left(e^{in\theta} + e^{in(2\phi-\theta)}\right) \mathrm{d}s$$

where $A^+_{\alpha}(\rho, \phi)$ is the part of arc corresponding to $\theta \ge \phi$. Further we observe that

$$e^{in\theta} + e^{in(2\phi-\theta)} = 2e^{in\phi}\cos n(\theta-\phi).$$

We therefore have

$$g^{\alpha}(\rho,\phi) = \sum_{n=-\infty}^{\infty} \int_{A^{+}_{\alpha}(\rho,\phi)} f_{n}(r) \cos[n(\theta-\phi)] e^{in\phi} \mathrm{d}s.$$

Comparing with Equation (4.3) we have

$$g_n^{\alpha}(\rho) = \int_{A_{\alpha}^+(\rho,\phi)} f_n(r) \cos[n(\theta - \phi)] \mathrm{d}s.$$
(4.4)

From Figure 4.2, a straightforward calculation gives

$$\theta - \phi = \arccos\left(\frac{r^2 + R^2 - \rho^2}{2rR}\right) \tag{4.5}$$

and

$$ds = \frac{r dr}{R \sqrt{1 - \left(\frac{\rho^2 + R^2 - r^2}{2\rho R}\right)^2}}.$$
(4.6)

Using Equations (4.5) and (4.6) in (4.4) we get

$$g_n^{\alpha}(\rho) = \int_{R-\rho}^{\sqrt{R^2 + \rho^2 - 2R\rho \cos \alpha}} \frac{r \cos \left(n \cos^{-1} \left(\frac{r^2 + R^2 - \rho^2}{2rR}\right)\right) f_n(r)}{R\sqrt{1 - \left(\frac{\rho^2 + R^2 - r^2}{2\rho R}\right)^2}} dr$$
(4.7)

Letting $\cos(n \arccos x) = T_n(x)$ and u = R - r, we have

$$g_n^{\alpha}(\rho) = \int_{R-\sqrt{R^2 + \rho^2 - 2\rho R \cos \alpha}}^{\rho} \frac{K_n(\rho, u)}{\sqrt{\rho - u}} F_n(u) \mathrm{d}u$$
(4.8)

where

$$F_n(u) = f_n(R-u)$$

and

$$K_n(\rho, u) = \frac{2\rho(R-u)T_n\left[\frac{(R-u)^2 + R^2 - \rho^2}{2R(R-u)}\right]}{\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}}.$$
(4.9)

4.2.1 Function Supported Outside Acquisition Circle

Next we consider the reconstruction of functions supported outside the acquisition circle. More precisely, we consider functions supported inside the annular region $A(R_1, R_2)$ where $R_1 = R$ is the inner radius and $R_2 = 3R$ is the outer radius of the annulus. R is the radius of the acquisition circle P. The acquisition setup for this case is illustrated in Figure 4.3.

A similar derivation as above leads to the following Volterra integral equation of the first kind: $R_{\pm a}$

$$g_n^{\alpha}(\rho) = \int_{\sqrt{R^2 + \rho^2 + 2\rho R \cos \alpha}}^{R+\rho} \frac{rT_n(\frac{R^2 + r^2 - \rho^2}{2rR})}{\sqrt{1 - \left(\frac{R^2 + \rho^2 - r^2}{2\rho R}\right)^2}} f_n(r) \,\mathrm{d}r.$$
(4.10)

Substituting u = r - R we have

$$g_n^{\alpha}(\rho) = \int_{\sqrt{R^2 + \rho^2 + 2\rho R \cos \alpha}}^{\rho} \frac{F_n(u) \cdot K_n(\rho, u)}{\sqrt{\rho - u}} du$$
(4.11)

where $F_n(u) = f(R+u)$ and

$$K_n(\rho, u) = \frac{2\rho(R-u) \cdot T_n(\frac{(R-u)^2 + R^2 - \rho^2}{2R(R-u)})}{\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}}$$

Note that in this case, the kernel of the integral transform is the same as in Equation (4.8), but, as is to be expected, the limits of the integral are different.

The analogue of Equations (4.8) and (4.11) arising in full circular Radon transform are Volterra integral equations of first kind, where one of the limits is fixed. These were studied in [59, 60]. An exact solution of such equations arising in full circular Radon transform is known. However, the exact solution is numerically unstable. An efficient numerical algorithm for the inversion of Volterra integral equations of the first kind appearing in [59, 60] recently appeared in [70]. In the case under consideration, however, both the limits of integration are variable, and an exact inversion formula in not known to the best of our knowledge. Instead, following the algorithm given in [70], we present an efficient numerical inversion method to deal with the inversion of such nonstandard Volterra integral equations of the first kind. The presence of edges of the circular arcs in the domain introduces artifacts in the reconstructed images. Furthermore, the fixed angular span α places restrictions on the edges that are visible, leading to a streaklike artifacts. We propose an artifact suppression strategy that reduces some of these artifacts in this paper. To invert the transform, we directly discretize Equation (4.8)and invert using a Truncated Singular value Decomposition (TSVD); a method originally proposed in [81]. In the next section, we explain the numerical inversion algorithm as well as a method for the suppression of artifacts.

4.3 Numerical Inversion

4.3.1 Forward Transform

The forward transform is computed by discretizing Equation (4.1). It may be noted that we consider only partial data in the radial variable. The discrete transform is computed



FIGURE 4.2: Setup for functions supported inside the acquisition circle.

for $\rho \in [0, R - \epsilon_{\rho}], \epsilon_{\rho} > 0$. We have

$$g^{\alpha}(\rho_k, \phi_p) = \sum_{(x_n, y_m) \in \mathcal{A}_{k,p}} f(x_n, y_m), \qquad (4.12)$$

where

$$\mathcal{A}_{k,p} = \left\{ (x_n, y_m) : \sqrt{(x_n - R\cos\phi_p)^2 + (y_m - R\sin\phi_p)^2} = \rho_k^2 \\ ,\phi_p - \alpha \le \arctan(\frac{y_m}{x_n}) \le \phi_p + \alpha \right\},$$
$$\rho_k = kh, k = 0, 1, \dots, M - 1, \quad h = \frac{R - \epsilon_\rho}{M},$$

and

$$\phi_p = pl, j = 0, 1, \dots, N - 1, \quad l = \frac{2\pi}{N}.$$

Note that $g^{\alpha}(\rho_k, \phi_p)$ is an $M \times N$ matrix.

Figure 4.4 shows an image f(x, y) and the corresponding CAR transform g^{α} for $\alpha = 25^{\circ}$ and M = N = 300.



FIGURE 4.3: Setup for functions supported outside the acquisition circle



FIGURE 4.4: Sample image and corresponding Circular arc Radon transform for $\alpha = 25^{\circ}$. The dotted circle surrounding the image represents the acquisition circle.

4.3.2 Computation of Fourier Series

Given the data matrix $g^{\alpha}(\rho_k, \phi_p)$, we compute the discrete Fourier series coefficients g_n^{α} using the FFT algorithm. We assume the matrix $g^{\alpha}(\rho_k, \phi_p)$ to be real. Note that g_n^{α} is a vector of length M given by

$$g_n^{\alpha}(\rho_k) = \sum_{p=0}^{N-1} g^{\alpha}(\rho_k, \phi_p) \cdot e^{-i2\pi n \frac{p}{N}}$$

4.3.3 Computation of forward transformation matrix

Equation (4.8) can be discretized and written in the matrix form as follows

$$g_n^{\alpha} = B_n F_n \tag{4.13}$$

where

$$g_n^{\alpha} = \begin{pmatrix} g_n^{\alpha}(\rho_0) \\ \vdots \\ \vdots \\ g_n^{\alpha}(\rho_{M-1}) \end{pmatrix} \quad F_n = \begin{pmatrix} F_n(\rho_0) \\ \vdots \\ \vdots \\ F_n(\rho_{M-1}) \end{pmatrix}$$

Matrix B_n is a piecewise linear, discrete approximation of the integral kernel in Equation (4.8), g_n^{α} is the Fourier series coefficients of the circular arc Radon data and F_n the Fourier series coefficients of the original unknown function. The matrix B_n is computed using the trapezoidal rule [71, 72]. The method essentially breaks the full integral into a sum of M integrals. The function is approximated to be linear in each interval so that

$$\sqrt{h} \left\{ \sum_{q=l}^{k} b_{kq} K_n(\rho_k, \rho_q) F_n(\rho_q) \right\} = g_n(\rho_k)$$
(4.14)

where

$$b_{kq} = \begin{cases} \frac{4}{3} \{ (k-q+1)^{\frac{3}{2}} + \frac{4}{3} (k-q)^{\frac{3}{2}} + 2(k-q)^{\frac{1}{2}} & q = l \\ \frac{4}{3} \left((k-q+1)^{\frac{3}{2}} - 2(k-q)^{\frac{3}{2}} + (k-q-1)^{\frac{3}{2}} \right) & q = l+1, \dots k-1, \\ \frac{4}{3} & q = k. \end{cases}$$

and $l = \max\left(0, \left\lfloor R - \sqrt{R^2 + \rho_k^2 - 2\rho_k R \cos \alpha} \right\rfloor\right)$ where $\lfloor x \rfloor$ is the greatest integer less than equal to x.

The detailed derivation of Equation (4.14) is given in Appendix A. From Equation (4.14) it is clear that the entries of matrix B_n are independent of both the data $g^{\alpha}(\rho_k, \phi_p)$ as

well as the function f to be recovered. Hence, the matrix B_n may be pre-computed and stored.

Remark 4.1. The solution of inverse discrete circular arc Radon transform exists, and is unique.

From equation (4.14) we have,

$$[B_n]_{kk} = \frac{4}{3}\sqrt{h} \neq 0$$

also, $[B_n]_{kq} = 0 \ \forall \ q > k$
 $\Rightarrow \det(B_n) = \left(\frac{4}{3}\sqrt{h}\right)^M \neq 0$

Therefore solution F_n of Equation (4.13) exists and is unique. While the B_n matrix is invertible, in practice, it is ill-conditioned. In order to obtain a numerically stable inverse, we use Truncated Singular Value Decomposition (TSVD) based pseudo-inverse of the matrix. The TSVD based method is explained briefly in the next section.

4.3.4 Inversion using Truncated Singular Value Decomposition

TSVD is a commonly used technique to compute the pseudo-inverse of matrices. This method was introduced in [81] as a numerically stable method for solving least squares problem. The method involves the following steps.

- 1. Consider the singular value decomposition of matrix B_n such that $B_n = UD_nV^T$. D_n is an $n \times n$ diagonal matrix of singular values of B_n and U, V are orthogonal matrices consisting of left and right singular vectors of B_n respectively.
- 2. A rank r approximation $B_{n,r}$ of the matrix B_n , is given by $B_{n,r} = UD_rV^T$, where D_r is a diagonal matrix with

$$D_r(i,i) = \begin{cases} D_n(i,i) = \sigma_i, & i \le r \\ 0 & i > r. \end{cases}$$

3. Then the rank r inverse of the matrix is given by $B_{n,r}^{-1} = V D_r^{-1} U^T$ where,

$$D_{i,i}^{-1} = \begin{cases} \frac{1}{\sigma_i} & i < r\\ 0 & \text{otherwise} \end{cases}$$

4. Using $B_{n,r}^{-1}$ in Equation 4.13 we have

$$F_n \approx B_{n,r}^{-1} g_n$$

5. The approximation of original function $f(r, \theta)$ may be obtained by computing inverse Fourier transform of F_n .

$$f(r_k, \theta_n) = \sum_{p=0}^{N-1} F_n(R - \rho_k, \phi_p) \cdot e^{i2\pi n \frac{p}{N}}.$$
(4.15)

The final image f(x, y) in the Cartesian coordinates is obtained by interpolating the the polar image $f(r_k, \theta_n)$ onto the Cartesian grid using bilinear interpolation. Algorithm 2 summarizes the steps involved in the numerical inversion.

Algorithm 2: Numerical Inversion Algorithm **Data:** Radon Transform, $g^{\alpha}(\rho, \phi)$ **Result:** $f(r, \theta)$ 1 Compute the Discrete Fourier series $g_n^{\alpha}(\rho)$, of input $g^{\alpha}(\rho, \phi)$ in the ϕ variable s.t. $g_n^{\alpha}(\rho_k) = \sum_{p=0}^{N-1} g^{\alpha}(\rho_k, \phi_p) \cdot e^{-i2\pi n \frac{p}{N}}$ for each n do $\mathbf{2}$ Compute $B_n = [a_{ij}K_{ij}^n]$ where, a_{ij} is given by equation 4.14, and $K_{ij}^n = K_n(\rho_i, \rho_j)$ 3 by equation (4.9)Let $B_{n,r} = UD_rV^T$, with $D_r = diag(\sigma_1, ..., \sigma_r)$, s.t SVD of $B_n = UD_nV^T$, Compute low rank inverse, $B_{n,r}^{-1} = VD_r^{-1}U^T$ 4 $\mathbf{5}$ $F_n = B_{n,r}^{-1} \ g_n^{\alpha}$ 6 $\mathbf{7}$ end Compute inverse Fourier transform f using Equation (4.15). 8

4.4 **Results And Observations**

The following figures (Figure 4.6, 4.7 and 4.8) show samples results of reconstruction with different opening angles α for phantoms shown in figure 4.5. From the samples results following observations may be made.

- *Effect of opening angle:* We observe that for small opening angle the reconstruction quality is not good. However, as the opening angle increases, the quality of reconstructed image improves as well.
- *Ringing Artifacts:* We observe a presence of ringing artifacts in the reconstructed images. These artifacts are in the form of circular rings centered at the origin. Additionally, there is a sharp circular artifact, whose size increases with opening angle α.



FIGURE 4.5: Phantoms used in the experiments



FIGURE 4.6: Example image reconstructions using algorithm described in Algorithm 2 for phantom 1 shown in figure 4.5

In order to improve the quality of reconstruction we propose a method for reduction of artifacts. The details of the same are discussed in the next chapter.

 $(A) \alpha = 6$ $(B) \alpha = 16$ $(C) \alpha = 21$ $(C) \alpha = 21$ $(C) \alpha = 21$ $(C) \alpha = 31$ $(E) \alpha = 46$ $(F) \alpha = 76$

FIGURE 4.7: Example image reconstructions using algorithm described in Algorithm 2 for phantom 2 shown in figure 4.5



FIGURE 4.8: Example image reconstructions using algorithm described in Algorithm 2 for phantom 3 shown in figure 4.5

Chapter 5

Analysis and Suppression of Artifacts

"There is nothing more deceptive that an obvious fact"

-Sherlock Holmes, The Boscombe Valley Mystery

5.1 Introduction

In the classical setting of the tomographic reconstruction problem, the tomographic data $(= \mathcal{R}f(\phi, \rho))$ is assumed to be known for all values $(\phi, \rho) \in S^1 \times \mathbb{R}$. This tomographic problem has been extensively studied in both linear as well as circular case and reconstruction algorithms are available for complete data, see for example [60]. However, in the present work we consider data which is limited in both ϕ as sell as ρ variable, i.e tomographic data is not available for the whole $S^1 \times \mathbb{R}$. This *limited data* scenario introduces singularities in the acquired data due to hard cut off of the data. These singularities in the data manifest in the form of artifacts in the reconstructed image. A framework for analysing and handling artifacts has been provied in [77]. The framework provides a general approach for analysing the artifacts in computed tomography. In the present chapter we analyse, numerically, the effect of various parameters on the artifacts in the reconstructed image. Further, based on [77] we propose an artifact suppression strategy for the Circular arc Radon transform.

5.2 Artifacts in numerical reconstructed images

We use the strategy discussed in Section 4.3 to reconstruct analytical phantoms shown in Figure 5.1. Numerical reconstruction results for analytical phantoms exhibit different



FIGURE 5.1: Phantoms used in the experiments

kinds of artifacts. While some artifacts are inherent to the problem, others arise due to numerical issues. Various parameters which affect the artifacts include, opening angle α , rank of matrix $B_{n,r}$, discretization of angular and radial variables and numerical approximation. In this section, we discuss the effects of some of the parameters on the artifacts and in the following section (Section 5.3), we propose a modified algorithm for suppression of the artifacts.

5.2.1 Effect of Rank of $B_{n,r}$ matrix

During reconstruction, the view angle α will be determined by the transducer, while the discretisation of angular and radial variables are chosen as part of an imaging protocol depending on constraints on acquisition time, sensor bandwidth and sensitivity.

At the algorithm level, a key parameter affecting the quality of reconstruction is the rank r of the matrix $B_{n,r}$. The matrix B_n is non-singular, however due to the high condition number $(\mathcal{O}(10^{15}))$, a full rank (r = n) inversion will be unstable and will not result in a meaningful reconstruction. Therefore, an r-rank (with r < n,) approximation of the matrix B_n is one approach to stable inversion. Such a low-rank approximation is achieved in the proposed numerical scheme via TSVD.

SVD decomposes a signal f into a sum of harmonics $f = \sum_{i=1}^{n} \sigma_i u_i v_i^T$. Consequently, setting $\sigma_i = 0$ for i > r in the TSVD of $B_{n,r}$ will lower the number of harmonics in the reconstructed image leading to ringing artifacts. Figure 5.2 shows reconstruction for a fixed view angle $\alpha = 31^{\circ}$ with different r. The results are as expected, with good quality reconstruction seen for r = 0.9n and visible degradation in the quality with a reduction in r. Specifically, severe ringing artifacts can be seen in the result when r = 0.5n or



FIGURE 5.2: Effect of rank r of matrix $B_{n,r}$ on the reconstruction quality. n = 300 in all the above examples

lower. Thus, there is a tradeoff between rank and quality of reconstruction. Figure 5.3 shows a similar relation for the case of object supported outside the acquisition circle.

5.2.2 Effect of opening angle α on quality of reconstructed images.

Despite the fact that the view angle is a parameter that is generally fixed for a particular imaging setup, it is of interest to gain insight into the relationship between this parameter and the quality of reconstruction. In general, limiting the view by restricting the angular span α should introduce artifacts, as all edges in the object may not be visible. This notion of visibility can be explained as follows. If the data set C representing the curves of integration, are smooth objects such as lines, full circles, spheres etc., then roughly speaking, for an edge in the image to be stably reconstructed, there should be an element of C tangential to the edge. A formal justification of this statement is possible with the tools of Fourier integral operator theory and microlocal analysis[56, 57, 82]. We refer to all edges which are tangential to the interiors of the arcs in the data set C, as visible edges. Edges which fail to satisfy this condition are not reconstructed stably. Such a principle, for our set up, can be applied for edges at points where the interiors of the arcs satisfy the aforementioned tangential condition. However, due to the corners



FIGURE 5.3: Effect of rank r of matrix $B_{n,r}$ for support outside case. n = 300 in all examples

of the arcs inside the domain, we expect artifacts to be present in the reconstructed image. A rigorous study of the microlocal analysis of CAR transform, in particular, the characterization of the added artifacts into the reconstructed image is an important and challenging problem and we hope to address this in a future work. Figure 5.4 shows reconstruction at r = 0.9n for various α . We observe from these results that, as expected, the reconstruction of the visible edges is sharp, whereas the other edges are blurred out. As the angle α increases, the visible region of edges increases, and hence most of the edges in the images with large α are reconstructed. Larger α corresponds to a wider arc, and therefore more edges are tangential to the curve of integration. This dependence on α is clearly observed in the lower ellipse in Figure 5.4. As the span of arc increases, some arcs become tangential to the lower boundary of the ellipse. Hence we observe that the lower portion of the ellipse becomes sharper as α increases. A similar behavior is also observed in Figure 5.5 where the support of the function is outside the acquisition circle. In this Figure, there are circular arcs in the data set tangential to the edges in a neighborhood of the radial direction whereas none is tangential in the complement of such directions. Therefore, these edges are blurred out and the reconstruction of the edges does not appear to improve with increasing α .

We also observe various streaks and a strong circular artifact whose location changes with α . These artifacts are to be expected as we are dealing with a limited view problem. Handling of these artifacts to improve the quality of reconstructed image is considered



FIGURE 5.4: Effect of change in α on the visible edges.

in the next section. A similar behavior is also observed in the case where the support of the function is outside the acquisition circle. Figure 5.5 shows the reconstruction results for various values of α in this case.

5.3 Suppression of Artifacts

To understand the source of artifacts, and subsequent suppression in the reconstructed images we re-write the CAR transform (equation 4.1) as follows.

$$g^{\alpha}(\rho,\phi) = \mathcal{R}_{\alpha}f(\rho,\phi) = \int_{C(\rho,\phi)} \chi_A f(r,\theta) \,\mathrm{d}s, \qquad (5.1)$$

where, χ_A is the characteristic function of the arc $A_{\alpha}(\rho, \phi)$ such that

$$\chi_A = \begin{cases} 1, & (r,\theta) \in A_\alpha(\rho,\phi) \\ 0, & \text{else} \end{cases}$$



FIGURE 5.5: Effect of change in α in case of support outside acquisition circle.



FIGURE 5.6: Location of sharp artifact(red circle) with respect to the arcs(orange)

The function χ_A truncates the full data $f(r,\theta)$ before computation of the integral. χ_A is a Heaviside type function with sharp cut-off at the edges of the arc. Since the data is measured only for $\rho \in (0, R - \epsilon_{\rho})$, there is also a similar Heaviside type truncation function in the radial direction, with hard truncation at $\rho = R - \epsilon_{\rho}$.

It should be noted that $\chi_A \equiv 1$ for the whole circle $C(\rho, \phi)$ in the circular Radon transform while, $\chi_A = 0$ beyond the arc $A_{\alpha}(\rho, \phi)$ in the CAR transform. The sharp truncation in data should lead to strong streaking artifacts in the reconstructed result. Moreover, a strong circular artifact is also observed along the edges of arcs corresponding to largest value of ρ as depicted in figure 5.6. The double penalization in the form of hard truncation of data i) at the edges of the arc in the angular direction and ii) in the radial direction for $\rho = R - \epsilon_{\rho}$, we believe, is the reason for the sharp circular artifact at a specific radial location. In order to suppress the artifacts, we modify the characteristic function to $\hat{\chi}_A$, so that it decays smoothly instead of going to 0 abruptly at the edge. This smooth decay serves to remove the singularity due to the Heaviside-type truncation. We choose a smooth, squared exponential decay of the form $e^{\frac{-x^2}{\sigma^2}}$. Specifically, the values of the matrix B_n are weighted by an exponential decay factor of the form $e^{\frac{-(j-m)^2}{\sigma^2}}$ as explained in algorithm 3. Here, σ controls the degree of smoothing. A large σ results in excessive smoothing and hence lead to blurring of the true edges of the reconstructed image. A low σ results in minimal smoothing, preserving edge definition but also in retention of artifacts. In our experiments, we chose $\sigma = 40$ which suppresses the strong streak artifacts, (see figure 5.4), whilst retaining the definition of true edges in the image. Algorithm 2 is a modified version of the numerical inversion Algorithm 1 and includes artifact suppression with $\hat{\chi}_A$.

As noted in Section 4.3, matrix B_n is a lower triangular matrix. Figure 5.7 is a visualization (as an image) of the structure of matrix B_n in the original and the modified form. Here, the white/black pixels indicate non-zero/zero entries. The modification of the transformation matrix leads to a slow decay of numerical values as shown in Figure 5.7 with grey coloured pixels. This helps smooth the sharp circular artifacts generated in the inversion process. Note that only the matrix B_n , which is constant for a given setup, is changed in the modified algorithm. The data, $g^{\alpha}(\rho, \phi)$ is not changed or pre-processed in any form. Figure 5.4, 5.5 show the reconstructed images after artifact suppression is performed for the cases of function supported inside and outside, respectively. We observe that using the modified algorithm, the sharp circular artifacts are significantly suppressed while the true edges of the image are retained.



FIGURE 5.7: Structure of matrix B_n . The original matrix (left) has a sharp cut off in the entries of B_n , while in the modified matrix (right) they decay smoothly.



FIGURE 5.8: Results of modified algorithm.

Algorithm 3: Numerical Inversion Algorithm **Data:** Radon Transform, $q^{\alpha}(\rho, \phi)$ **Result:** $f(r, \theta)$ 1 Compute the Discrete Fourier series $g_n^{\alpha}(\rho)$, of input $g^{\alpha}(\rho, \phi)$ in the ϕ variable s.t. $g_n^{\alpha}(k) = \sum_{p=0}^{N-1} g^{\alpha}(k,p) \cdot e^{-j2\pi n \frac{p}{N}}$ $\mathbf{2}$ for each n do Compute $B_n = [\hat{b_{ij}} K_{ij}^n]$ where $\hat{b_{ij}} = \gamma_A b_{ij}$ s.t., 3 $\hat{b_{ij}} = \begin{cases} e^{\frac{(i-j)^2}{\sigma^2}} b_{ij}, & j < h \\ b_{ij}, & h \le j \le i \end{cases}$ where b_{ij} is given by equation (4.14) with l = 0, $h = max\left(0, \left\lfloor R - \sqrt{R^2 + \rho^2 - 2\rho R \cos \alpha} \right\rfloor\right)$ and $K_{ij}^n = K_n(\rho_i, \rho_j)$ given by equation (4.9)Let $B_{n,r} = UD_rV^T$, with $D_r = diag(\sigma_1, ..., \sigma_r)$, s.t SVD of $B_n = UD_nV^T$, Compute low rank inverse, $B_{n,r}^{-1} = VD_r^{-1}U^T$ $\mathbf{4}$ 5 $F_n(\rho) = B_{n,r}^{-1} g_n^{\alpha}(\rho)$ 6 $\mathbf{7}$ \mathbf{end}

8 Compute inverse Fourier transform, $f(r, \theta) = \mathfrak{f}_{\phi}^{-1}(F(\phi_n, \rho))$

5.4 Conclusion

We presented a numerical algorithm to invert circular arc Radon transforms (CAR) arising in some imaging applications. The numerical algorithm required the solution of ill-conditioned matrix problems which was accomplished using a TSVD method. The method is efficient and the pipeline can be preprocessed to improve the speed. Compared to the inversion of full circular Radon transform, the quality of image reconstruction in the case of CAR is poorer due to the following reasons.

- a) The edges of the arcs of circles introduce strong artifacts.
- b) Several edges of the image are invisible due to the limit in the angular span of the arcs.

These lead to some streak artifacts as well. We presented a numerical algorithm to invert and supress these artifacts. The Artifact suppression algorithm proposed significantly reduced the strength of the artifacts while retaining the the strength of original image edges.



FIGURE 5.9: Results for artifact suppression in support outside case.

Chapter 6

Conclusion And Future Work

The woods are lovely, dark and deep, But I have promises to keep, And miles to go before I sleep ...

-ROBERT FROST, Stopping By Woods On A Snowy Evening

Tomographic image reconstruction, due to its numerous real world applications, is an extensively studied problem. The study of reconstructing a function from its projections/line integrals, is at least a century old [83]. It was shown in [84], that infinitely many projections are required to reconstruct an image perfectly. It was understood that incomplete data would leads to artifacts in the reconstructed images. In addition to the artifact problem, arising due to limited data availability, the reconstructed images are known to be corrupted by noise. In many modalities, especially nuclear imaging modalities like PET, this problem is typically severe.

In this thesis, we discussed these two problems in two different setups. To mitigate the noise problem we propose a change of reconstruction lattice from square to hexagonal. We also study image reconstruction from Radon transform along arcs of fixed angular span, thus proposing a new variant of Radon transform. We give a numerical inversion method for the transform, and also propose algorithm for reduction of artifacts generated in this case. Based on our study the following conclusions may be drawn.

i) Image reconstruction and denoising for Hexagonal lattices. We considered linear Radon transform and propose reconstruction and denoising on hexagonal lattices based on Filtered Back Projection for reduction of noise and improvement of quality of reconstructed image. We observed that there is both qualitative as well as quantitative improvement in the reconstructed image. Such a method may not me feasible for natural images, acquired directly in spatial domain. The resampling of images may lead to losses in such a case. However, in case of reconstructed/computed images using a hexagonal lattice is a better option. With the increasing use of computed images, such a change in lattice may be a significant source of qualitative improvement in images.

- ii) Circular arc Radon transform. We proposed a new Circular Arc Radon(CAR) transform, a generalization of full circular Radon transform. This transform is a generalization in the sense that we consider data only for a limited angular region of the circle. We also proposed a back projection based approximate inversion algorithm method for the same. The back-projection based method lead to a blurry reconstruction, typical of back projection algorithm. While post-processing in the form of hight pass filtering did improve the reconstruction quality to some extent, the reconstructed image is not sharp.
- iii) Fourier Series based inversion of CAR Transform. We numerically inverted CAR transform using the Fourier series method and discussed different parameters affecting the quality of reconstructed image. The method is based on the classic method proposed by Cormak[63]. We use a TSVD based regularization method to deal with the numerical instabilities. The method is fast and leads to a sharp reconstruction, however, due to limited data, arifacts are observed in the reconstructed images.
- iv) Arifact suppression algorithm for Fourier series based inversion. Due to the partial data acquired in CAR transform, a lot of artifacts are observed in the reconstructed image. We proposed an algorithm for suppression of artifacts which arise in the reconstruction process. The algorithm helps to reduce the strength of artifacts without affecting the original object. While we have tested the algorithm for CAR transform, it may be used to reduce similar artifacts which arise in similar limited data problems such as broken ray transform.

6.1 Future Work

The following are the possible directions of future work.

• Providing an analytical explanation for improved quality for reconstruction on hexagonal lattice: The improvement of denoising performance on hexagonal lattice has not been explained analytically. A possible future work could be to explain analytically, the reason for such an improvement.

A possible approach to the problem: The analysis may be approched by considering the manner in with hexagonal lattice discretizes the euclidean space. One may consider 2 systems, namely, a square discretizer and a hexagonal discretizer and show that there is more information loss in the case of square discretization.

• Closed form solution to the non-standard Volterra equations arising in CAR Transform: The theoretical inversion of CAR leads to some very interesting non-standard Volterra integral equations of the first kind. The Volterra integral equation where both limits are functions, with weakly singular kernel, which arise in such a transform, have not been dealt with in the current literature. A possible future extension of the present work may be to solve this problem.

A possible approach to the problem: The problem may be approached by extending the integral over the arc to full circle with the kernel set to zero for rest of the circle and consider the full inverse. However, due to the non-smooth nature of the kernel, inversion in such a case may not be straight forward.

• *Micro-local Analysis of Artifacts:* While the artifact suppression algorithm has been proposed based on intuitive understanding of the micro-local aspects of the Radon Transform, a rigorous analysis has not been presented in this work. Rigorous micro-analysis of the Circular arc Radon Transform has not been reported in literature and may be attempted as part of the future work.

A possible approach to the problem: The Micro-local analysis of the transform may be done using the framework provided in [75]. The analysis will involve calculating the canonical relation for the transform and analysing its action on the wavefront set of the function to be transformed.

Appendix A

Trapezoidal integration

The Fourier coefficients of the forward transform are given by equation 4.8 restated below.

$$g_n^{\alpha}(\rho) = \int_{R-\sqrt{R^2 + \rho^2 - 2\rho R \cos \alpha}}^{\rho} \frac{F_n(u)K_n(\rho, u)}{\sqrt{\rho - u}} \mathrm{d}r.$$

The integral is approximated by the sum in Equation (4.14). The sum is obtained by the trapezoidal product integration method proposed in [71, 72] (see also [70]) which we briefly outline below.

Let M be a positive integer and $\rho_l = lh$, l = 0, ..., M and $h = \frac{R-\epsilon}{M}$ be a discretization of radial variable $\rho \in [0, R - \epsilon]$ The above equation may be rewritten as follows

$$g_n^{\alpha}(\rho_k) = \sum_{q=1}^k \int_{\rho_{q-1}}^{\rho_q} \frac{F_n(u)K_n(\rho, u)}{\sqrt{\rho - u}} \mathrm{d}u.$$

We approximate $F_n(u) \cdot K_n(\rho, u)$ by linear function in the interval $[\rho_{q-1}, \rho_q]$, such that

$$F_n(u)K_n(\rho, u) \approx F_n(\rho_{q-1})K_n(\rho_k, \rho_{q-1})\frac{\rho_q - u}{h}$$
$$+F_n(\rho_q)K_n(\rho_k, \rho_q)\frac{u - \rho_{q-1}}{h}.$$

Here the function takes values $F_n(\rho_{q-1})K_n(\rho_k, \rho_{q-1})$ and $F_n(\rho_q)K_n(\rho_k, \rho_q)$ at the end points of the interval respectively. Hence we have

$$g_{n}^{\alpha}(\rho_{k}) = \sum_{q=1}^{k} \int_{\rho_{q-1}}^{\rho_{q}} \frac{1}{\sqrt{\rho - u}} \Big[F_{n}(\rho_{q-1}) K_{n}(\rho_{k}, \rho_{q-1}) \frac{\rho_{q} - u}{h} + F_{n}(\rho_{q}) K_{n}(\rho_{k}, \rho_{q}) \frac{u - \rho_{q-1}}{h} \Big] \mathrm{d}u.$$

Simple integration gives

$$h^{\frac{-3}{2}} \int_{\rho_{q-1}}^{\rho_q} \frac{\rho_q - u}{\sqrt{\rho_k - u}} du = -\frac{4}{3} \{ (k - q + 1)^{\frac{3}{2}} - (k - q)^{\frac{3}{2}} \} + 2(k - q + 1)^{\frac{1}{2}}$$

and,

$$h^{\frac{-3}{2}} \int_{\rho_{q-1}}^{\rho_q} \frac{u - \rho_{q-1}}{\sqrt{\rho_k - u}} du = \frac{4}{3} \{ (k - q + 1)^{\frac{3}{2}} - (k - q)^{\frac{3}{2}} \} - 2(k - q)^{\frac{1}{2}}.$$

Hence we have

$$g_n^{\alpha}(\rho_k) = \sqrt{h} \sum_{q=l}^k \left[-\frac{4}{3} \{ (k-q+1)^{\frac{3}{2}} - (k-q)^{\frac{3}{2}} \} + 2(k-q+1)^{\frac{1}{2}} \right]$$
$$\times F_n(\rho_{q-1}) K_n(\rho_k, \rho_{q-1})$$
$$+ \left[\frac{4}{3} \{ (k-q+1)^{\frac{3}{2}} - (k-q)^{\frac{3}{2}} \} - 2(k-q)^{\frac{1}{2}} \right]$$
$$\times F_n(\rho_q) K_n(\rho_k, \rho_q).$$

From the support assumption, we have $F_n(p) = 0 \ \forall p \leq 0$. Then the above expression simplifies to the following expression.

$$\sqrt{h}\left\{\sum_{q=l}^{k} b_{kq} K_n(\rho_k, \rho_q) F_n(\rho_q)\right\} = g_n(\rho_k)$$

where

$$b_{kq} = \begin{cases} \frac{4}{3} \{ (k-q+1)^{\frac{3}{2}} + \frac{4}{3} (k-q)^{\frac{3}{2}} + 2(k-q)^{\frac{1}{2}} & q = l \\ \frac{4}{3} \left((k-q+1)^{\frac{3}{2}} - 2(k-q)^{\frac{3}{2}} + (k-q-1)^{\frac{3}{2}} \right) & q = l+1, \dots k-1. \\ \frac{4}{3} & q = k. \end{cases}$$

and $l = \max\left(0, \left\lfloor R - \sqrt{R^2 + \rho_k^2 - 2\rho_k R \cos \alpha} \right\rfloor\right)$ where $\lfloor x \rfloor$ is the greatest integer less than equal to x.

Appendix B

Filtered Back Projection

Sinogram $p(r,\theta)$ is essentially a set of 1D projections (p(r)) of a 2D function f(x,y) taken at various angles (θ) . The basic problem in tomography is to 'invert' this process. The function $p(r,\theta)$ is known as the 'Radon transform' of the function f(x,y). $p(r,\theta)$ is the line integral of the image f(x,y) along the line l whose normal, angle of normal with respect to the x-axis is (r, θ) .

$$p(r,\theta) = \int_{l} f(x,y) dl$$

The equation of this line described above can be written as below

$$xsin\theta - ycos\theta = r$$

Therefore, the projection along line l can be rewritten as

$$p(r,\theta) = \int \int f(x,y)\delta(x\sin\theta - y\cos\theta - r)dxdy$$

This is the expression for Radon transform of function f(x, y). Given a sinogram $p(r, \theta)$ the problem of finding the original function f(x, y) can be expressed as

$$f(x,y) = \mathcal{R}^{-1} \left\{ p(r,\theta) \right\}$$

The above inverse is computed using the *central slice theorem*. Let $F(k_x, k_y)$ be the 2D Fourier transform of function f(x, y).

$$F(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi j (k_x x + k_y y)} dx dy$$

also, let $P_{\theta}(k)$ be the 1D Fourier transform of $p(r, \theta)$.

$$P_{\theta}(k) = \int_{\infty}^{-\infty} p(r,\theta) e^{-2\pi j k r} dr$$

for all values of θ we have then,

$$P(\theta, k) = F(k_x, k_y)$$

where,

$$k_x = k\cos\theta$$
$$k_y = k\sin\theta$$
$$k = \sqrt{k_x^2 + k_y^2}$$

Therefore, 1D Fourier transform of radon transform along r is equal to the 2D Fourier transform of the function. The original function f(x, y) can therefore be calculated if $P_{\theta}(r)$ is known for all values of $\theta \in (0, \pi)$.

From the central slice theorem stated above, we have

$$f(x,y) = \mathcal{F}^{-1}\{P(\theta,k)\}$$

where, \mathcal{F} denotes the inverse Fourier transform. Writing the above in polar form we have

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} P(k,\theta) |k| e^{2\pi j k r} dk d\theta$$

where $r = x\cos\theta + y\sin\theta$. Let

$$P^*(k,\theta) = P(k,\theta)|k|$$

Here, $P^*(k, \theta)$ represents the filtered (by |k|) version of the original projection in the Fourier domain. Inverse Fourier transform of this is found as

$$p^*(r,\theta) = \int_{-\infty}^{\infty} P^*(k,\theta) e^{2\pi j k r} dk$$

Finally, the desired image f(x, y) is found by back projecting $p^*(r, \theta)$ as

$$f(x,y) = \int_0^\theta p^*(r,\theta) d\theta$$

This algorithm is know as *filtered back projection* as it involves filtering of $P(k, \theta)$ by a ramp filter |k|
Related Publications

- PET Image Reconstruction And Denoising On Hexagonal Lattices.
 Syed T. A. and Sivaswamy J.
 International Conference on Image Processing(ICIP) 2015, Quebec city.
- Numerical inversion of circular arc Radon transform
 Syed T. A., Krishnan V. P. and Sivaswamy J. (Under review).
- Other Publications:

Latent Factor Model Based Classification for Detecting Abnormalities in Retinal Images.

Syed T. A. and Sivaswamy J.

Asian Conference on Pattern Recognition(ACPR) 2015, Kuala Lumpur.

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