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Vertex Magic Total Labelings of Complete Graphs *

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Abstract

A vertex magic total labeling of a graph $G = (V, E)$ is a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ such that for every vertex w, the sum $f(w) + \sum_{uw \in E} f(uw)$ is a constant. It is well known that all complete graphs K_n admit a vertex magic total labeling. In this paper we present a new proof of this theorem using the concepts of twin factorization and magic square.

Keywords: complete graphs, vertex magic total labeling, factorization.

2000 Mathematics Subject Classification: 05C78 – Graph labelling.

1. Introduction

Let $G = (V, E)$ be a finite, simple, and undirected. For graph theoretic terminology we refer to West [11]. We note a $m = |E|$ and $n = |V|$. The labeling of a graph is a map that takes graph elements V or E or $V \cup E$ to numbers (usually positive or non-negative integers). A comprehensive survey of graph labellings is given in Gallian [1]. The notion of vertex magic total labelling was introduced by Gray et al. [3, 4].

A vertex magic total labeling of a graph $G = (V, E)$ is a bijection f from $V \cup E$ to the set of integers $1, 2, 3, \cdots, n + m$ such that for every vertex v,

$$f(v) + \sum_{vw \in E} f(vw) = k, \text{ where } k \text{ is a constant.}$$

Miller, Macdougall, Slamin, and Wallis [7] gave a vertex magic total labeling of complete graphs K_n for $n \equiv 2 \pmod 4$ by making use of vertex magic total labeling of $K_{\frac{n}{2}}$, where $n/2$ is odd. Lin and Miller [6] gave a vertex magic total labeling of complete graphs K_n for $n \equiv 4 \pmod 8$ by making use of vertex magic total labeling of $K_{\frac{n}{4}}$, where $n/4$ is odd. Labellings for K_n, n odd, are presented by Macdougall, Miller, Slamin, and Wallis [8].

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A simpler proof of the fact that complete graphs have a vertex magic total labeling was given by Gray et. al [4].

Focusing on K_n, n odd, McQuillan and Smith [9], presented a new technique to arrive at a vertex magic total labeling of complete graphs of odd order for all values of k between $\frac{n(n^2+3)}{4}$ and $\frac{n(n^2+1)}{4}$. Our technique for K_n, n odd can be described by using their techniques.

More recently, Gomez [2] has shown that K_n has a super vertex magic labeling, a vertex magic labeling where the smallest labels are given to the vertices. A recent result of Gray shows that *all* regular graphs have a vertex magic total labeling [3].

Thus, it is clearly established that vertex magic total labeling exists for *all* complete graphs. In this paper we show, by using new techniques, that *every complete graph* K_n, $n \geq 1$, *has a vertex magic total labeling.* While our result is not new, the techniques are new and may be of independent interest. Our constructions are very simple, compared to the ones proposed in the literature [6] which uses mutually orthogonal Latin squares. Moreover, while our method is similar to that of [4] in some respects, the magic constant obtained by our method is smaller than the constant reported in [6, 4]. Further, the intuitive representation scheme makes the proposed method easy to understand.

We use ideas from *twin-factorizations* [10] and magic squares (cf. [5]) to build the desired labellings. In this direction, we first build a vertex magic total labeling of K_n when n is odd (cf. Theorem 2.1). This is then used to build a vertex magic total labeling of K_n when n is even by treating it in three cases: $n \equiv 2 \pmod 4$, $n \equiv 4 \pmod 8$, and $n \equiv 0 \pmod 8$. The labeling for even n is obtained by representing K_n as a union of three graphs: $2K_{n/2}$'s and a $K_{n/2,n/2}$. However, care must be taken when constructing the labels for the subgraphs of the original graph. This requires that (a) the set of labels 1 through $n + n(n-1)/2$ be partitioned, (b) the resulting partitions be used to label the three subgraphs independently, and (c) combine the labels of the subgraphs to obtain a vertex magic total labeling of the original graph, K_n.

1.1. A Note on Representation

To help us in visualizing the labels of the edges and the vertices, we use the matrix representation given in Figure 1 for the labels. We represent the vertex magic total labeling for K_n as an $n \times n$ matrix in which the entries of the first row were used to label the vertices of the graph. The remaining entries of the matrix were used to label the edges of the graph. The ith column of the matrix contains the labels of all the edges incident at vertex i, in rows 2 through n in the order $(v_i, v_1), (v_i, v_2), \cdots, (v_i, v_{i-1}), (v_i, v_{i+1}), \cdots, (v_i, v_m)$. Notice that in this matrix, each column sum shall be constant, called the *magic constant.*

The rest of the paper is organized as follows. In Section 2, we show the construction of the labeling for K_n, when n is odd. Section 3 shows the construction for the case when n is even. The paper ends with some concluding remarks in Section 4.

v_1	v_2	v_3	v_4	v_5	$\cdots$	v_n
(v_1, v_2)	(v_2, v_1)	(v_3, v_1)	(v_4, v_1)	(v_5, v_1)	$\cdots$	(v_n, v_1)
(v_1, v_3)	(v_2, v_3)	(v_3, v_2)	(v_4, v_2)	(v_5, v_2)	$\cdots$	(v_n, v_2)
(v_1, v_4)	(v_2, v_4)	(v_3, v_4)	(v_4, v_3)	(v_5, v_3)	$\cdots$	(v_n, v_3)
(v_1, v_5)	(v_2, v_5)	(v_3, v_5)	(v_4, v_5)	(v_5, v_4)	$\cdots$	(v_n, v_4)
.	.	.	.	.	.	.
.	.	.	.	.	.	.
(v_1, v_n)	(v_2, v_n)	(v_3, v_n)	(v_4, v_n)	(v_5, v_n)	$\cdots$	$(v_n, v_{(n-1)})$

Figure 1: The matrix of labels for K_n. The first row has the vertex labels and the ith column entries in rows 2 to $n-1$ have the edge labels for v_i, for $1 \leq i \leq n$.

2. Vertex magic total labeling for K_n where n is odd

Our construction of vertex magic total labeling of complete graphs K_n, where n is odd, is almost similar to construction of a magic square of order n. The construction of a magic square of order n is described in [5]. We however have to proceed differently as we do not use all the n^2 numbers as is the case with a magic square of order n. The details of our scheme are in the following theorem.

Theorem 2.1. *K_n, n odd, admits a vertex magic total labeling.*

Proof. It is trivial for $n = 1$. For $n > 1$, we construct the labeling as follows.

Consider an $n \times n$ matrix M. Let i, j be the indices for rows and columns respectively. The indices follow a zero based index. The process of filling the entries of the matrix is similar to constructing a magic square of order n. The only difference is that we stop filling the matrix once we have reached the number $n + \frac{n(n-1)}{2}$. The process is described below.

- **Step 1:** Let $x = 1$. Start populating the matrix from the position $j = \frac{(n-1)}{2}$, $i = \frac{(n+1)}{2}$.
- **Step 2:** Fill the current (i, j) position with the value x, and increment x by 1.
- **Step 3:** The subsequent entries are filled by moving south-east by one position(i.e. $i = (i+1) \bmod n, j = (j+1) \bmod n$), till a vacant entry is found. If the cell is already filled, then fill from the south-west direction ($i = i+1 \bmod n, j = j-1 \bmod n$). The value of x has to be incremented after filling an entry. If x reaches $n + \frac{n(n-1)}{2}$, then proceed to Step 4.
- **Step 4:** The remaining $n(n-1)/2$ entries are filled just by copying the non-diagonal entries by appealing to symmetry (i.e., just copying the numbers in the lower triangular matrix to the upper triangular matrix and vice-versa from filled cells to unfilled cells).

- **Step 5:** Rearrange the entries of the matrix so that the numbers placed along the principal diagonal are moved to the first row of the matrix and all the entires in the upper triangular portion of the matrix are moved down by one position.

□

Lemma 2.2. *The above labeling is a valid vertex total magic labeling of K_n, n odd. Moreover, the magic constant obtained is* $S(n) = \frac{n^4+3n^2}{4n}$.

Proof. Notice that the only difference from a magic square to our labeling matrix is that while in a magic square the entries range from 1 to n^2, here we have used only labels from 1 to $n + n(n-1)/2$. The other entries can be seen to be reduced in magnitude by an amount of $n(n-1)/2$. Since this applies to all the entries, it can be seen that the properties of the magic square that all column sums are equal still applies in our case. Hence, the labeling obtained is a vertex magic total labeling of K_n, where n is odd.

It can be seen that the magic constant using the above construction is $S(n) = \frac{n^3+3n}{4}$. □

Example 2.3. *We provide an example below with* $n = 5$.

The matrices in Figure 2 show the result obtained after completion of Step 3 and Step 5.

11		07		03
04	12		08	
	05	13		09
10		01	14	
	06		02	15

Vertex label	11	12	13	14	15
Incident edges	04	04	07	10	03
	07	05	05	08	06
	10	08	01	01	09
	03	06	09	02	02

Figure 2: The resulting matrices after Step 3, on the left, and Step 5, on the right.

3. Vertex Magic Total Labeling for K_N, N even

In this section, we address the construction of vertex magic total labeling for K_n, when n is even. This is done in three steps. We treat the case where $n \equiv 2 \pmod 4$ (cf. Theorem 3.1 first as in that case $n/2$ is odd. This lets us use the result of Theorem 2.1 and compose labellings of smaller graphs to arrive at the labeling of K_n. This is then extended to the case where $n \equiv 4 \pmod 8$ and then finally where $n \equiv 0 \pmod 8$.

In the remainder of this section, the case of $n \equiv 2 \pmod 4$ is treated in Section 3.1, the case of $n \equiv 4 \pmod 8$ in Section 3.2, and the case of $n \equiv 0 \pmod 8$ is presented in Section 3.3.

3.1. Vertex Magic Total Labeling for K_N, $N \equiv 2 \pmod 4$

Theorem 3.1. *There is a vertex magic total labeling for K_N, for all $N \equiv 2 \pmod 4$.*

Proof. Let $n = N/2$. Our construction of vertex magic labeling for K_N makes use of a magic square of order n and a vertex magic total labeling of K_n. The construction of a magic square of order n and vertex magic total labelling for K_n is described in Section 2.

The basic idea behind constructing a vertex magic total labeling of these complete graphs is to view the graph K_N as a union of two complete graphs each of n vertices and a complete bipartite graph with n vertices on each side, i.e., $K_N = K_n \cup K_n \cup K_{n,n}$. We use the numbers 1 through $N + N(N-1)/2$ for the labels of the vertices and edges of K_N. To arrive at the labeling, we partition this set of numbers into five disjoint sets as follows.

$$
\begin{aligned}
S_1 &= \cup_{i=1}^{n-1/2}\{(2i+1)n+1, (2i+1)n+2, \cdots, (2i+2)n\} \cup \{1, 2, \cdots, n\}\\
S_2 &= \cup_{i=2}^{n-1/2}\{2i \cdot n+1, 2i \cdot n+2, \cdots, (2i+1)n\} \cup \{n+1, n+2, \cdots, 2n\}\\
S_3 &= \{2n+1, 2n+2, \cdots, 3n\},\\
S_4 &= \{n^2+n+1, n^2+n+2, \cdots, n^2+2n\},\\
S_5 &= \{n^2+1, n^2+2, \cdots, n^2+n\} \cup \{n^2+2n+1, n^2+2n+2, \cdots, 2n^2+n\}.
\end{aligned}
$$

In our construction, we use sets S_1 and S_4 for constructing the intermediate labeling L_1 for $K_{n/2}$. The elements of S_1 are used to label the edges and elements of S_4 are used to label the vertices. Similarly, we use sets S_2 and S_3 for constructing the intermediate labeling L_2 for another $K_{n/2}$. The elements of S_2 are used to label the edges and elements of S_3 are used to label the vertices. The elements of S_5 are used to label the edges in the bipartite graph $K_{n/2,n/2}$. An illustration of the construction is shown in Figure 3.

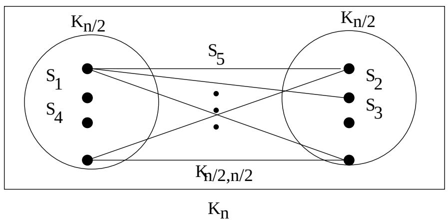

Figure 3: The construction of the labeling for K_N, with $N \equiv 2$ mod 4. The two $K_{n/2}$s and $K_{n/2,n/2}$ are shown along with the sets that are used to label the various components of K_n are also shown in the picture.

For constructing L_1 and L_2, first consider a vertex magic total labelling L for K_n. Since n is odd, this can be obtained using the construction given in Section 2. The matrices L_1 and L_2 are obtained from L as follows. Replace the vertex labels of L by the elements in S_4 and replace the edge labels by the elements of S_1, to get L_1. Similarly, if the edge labels are replaced by elements of S_2 and the vertex labels by S_3, we get L_2.

For constructing the labels of the edges in the bipartite graph $K_{n/2,n/2}$, consider a magic square of order $n/2$ and replace the elements of the magic square by the elements of S_5. Let us name this modified magic square as N_1. Transpose the magic square N_1 and name this as N_2. Finally, arrange all these intermediate labellings as shown below to get the labeling for K_N.

$$\begin{bmatrix} L_1 & L_2 \\ N_1 & N_2 \end{bmatrix}$$

It can be deduced that the magic constant in this case is $\frac{8n^3+6n^2-2n}{4}$.

Notice, however, that the above arrangement of labels needs to be adjusted to get the labeling matrix in the standard form described in Section 1.1. This rearrangement is required as the the first column of L_2 corresponds to edge labels $v_{n+1}v_{n+2}$, $v_{n+1}v_{n+3}, \cdots, v_{n+1}v_{2n}$ and the first column of N_2 corresponds to the edge labels of $v_{n+1}v_1$, $v_{n+1}v_2, \cdots, v_{n+1}v_n$. To get to the standard form, we need to shift the rows of N_2 by $n-1$ places up and shift the rows 2 through $n-1$ of L_2 by n places. □

Example 3.2. *Let us construct vertex magic total labelling for K_6 using a vertex magic total labelling of K_3. Let L below be the vertex magic total labeling of K_3, obtained by using the procedure of Section 2.*

vertex label	04	05	06
Incident edges	03	03	02
	02	01	01

From this L, L_1 is obtained by using elements of S_1 and S_4, with $S_1 = \{1, 2, 3\}$, and $S_4 = \{13, 14, 15\}$, we get the following matrix as L_1. Similarly, L_2 is obtained by using elements of S_2 and S_4 with $S_2 = \{4, 5, 6\}$ and $S_3 = \{7, 8, 9\}$.

$L_1 =$

Vertex label	13	14	15
Incident edges	03	03	02
	02	01	01

$L_2 =$

Vertex label	07	08	09
Incident edges	06	06	05
	05	04	04

To construct intermediate labelling N_1 and N_2, consider a magic square of order three as follows. In Figure 4, the matrix on the left shows a magic square of order three, the matrix on the right shows the magic square with elements replaced from elements in S_5.

04	09	02
03	05	07
08	01	06

16	21	11
12	17	19
20	10	18

Figure 4: The construction of N_1 from a magic square of order 3

The matrix N_2, for reasons of symmetry, is constructed as the transpose of N_1. The final matrix that shows a vertex magic total labeling of K_6, obtained by putting together L_1, L_2, N_1, and N_2, after the required rearrangement is shown in Figure 5.

Vertex label	13	14	15	7	8	9
Incident Edges	03	03	02	16	12	20
	02	01	01	21	17	10
	16	21	11	11	19	18
	12	17	19	06	06	05
	20	10	18	05	04	04

Figure 5: A vertex magic total labeling of K_6. The first row elements are the vertex labels. The column entries of the ith column give the edge labels of edges adjacent to vertex v_i. See also the representation in Figure 1.

3.2. Labeling for K_N where $N \equiv 4 \bmod 8$

We now move to the case where $N \equiv 4 \bmod 8$. We show the following theorem.

Theorem 3.3. *There is a vertex magic total labeling for K_N, for all $N \equiv 4 \bmod 8$.*

Proof. Our construction of vertex magic labeling for K_N makes use of magic squares of order $N/4$. Let $n = N/4$. The construction of a magic square of order n and a vertex magic total labelling for K_n is known to exist from Section . The basic idea behind constructing a vertex magic total labeling of K_N, $N \equiv 4 \bmod 8$, is to represent K_N as the union of four K_n's, two $K_{n,n}$s, and a $K_{2n,2n}$.

It is required to map the vertices and edges in these smaller graphs to numbers from 1 through $N + N(N-1)/2$. To this end, we partition the integers from 1 through $N + N(N-1)/2$ into eleven disjoint sets S_1 through S_{11} as follows.

$$S_1 = \cup_{i=1}^{n-1/2}\{(2i+1)n+1, (2i+1)n+2, \cdots, (2i+2)n\} \cup \{1, 2, \cdots, n\}$$
$$S_2 = \cup_{i=2}^{n-1/2}\{2i \cdot n+1, 2i \cdot n+2, \cdots, (2i+1)n\} \cup \{n+1, n+2, \cdots, 2n\}$$
$$S_3 = \{2n+1, 2n+2, \cdots, 3n\}$$
$$S_4 = \{n^2+n+1, n^2+n+2, \cdots, n^2+2n\}$$
$$S_5 = \{n^2+1, n^2+2, \cdots, n^2+n\} \cup \{n^2+2n+1, n^2+2n+2, \cdots, 2n^2+n\}.$$

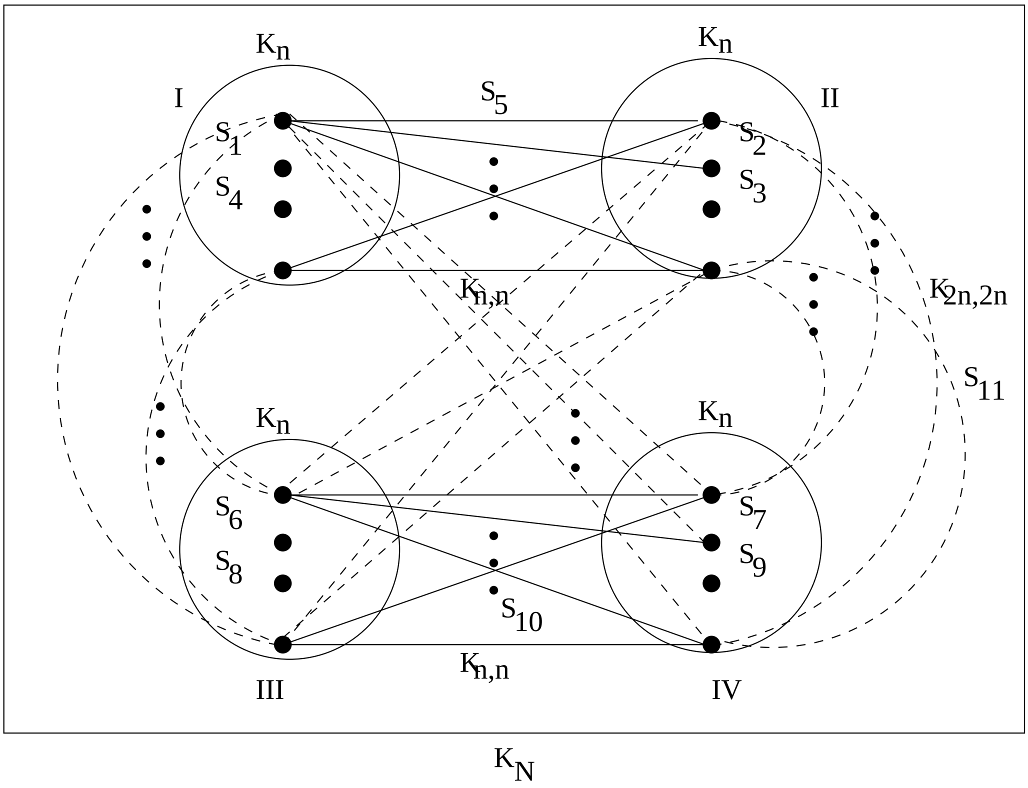

Figure 6: The construction of the labeling for K_N, with $N \equiv 4 \bmod 8$. The four smaller K_ns are labeled I, II, III, and IV. The sets that are used to label the various components of K_N are also shown in the picture. The dashed edges correspond to $K_{2n,2n}$.

The sets S_6, S_7, S_8, S_9, and S_{10} are constructed by adding $2n^2 + n$ to each element of S_1 through S_5 respectively. Finally, $S_{11} = \{4n^2 + 2n + 1, 4n^2 + 2n + 2, \cdots, 8n^2 + 2n\}$ will be a set of $4n^2$ elements.

The way we use these sets is as follows. We use sets S_1 and S_4 for constructing intermediate labeling L_1 for a K_n with the elements of S_1 used to label the edges and elements of S_4 for labeling the vertices. Similarly, for labeling another K_n, we use the elements of S_2 to label the edges and the elements of S_3 to label the vertices. For the third K_n, we use the elements of S_6 to label the edges and the elements of S_8 to label the vertices. Similarly, for the fourth K_n, we use the elements of S_7 to label the edges and the elements of S_9 to label the vertices. In the above, the terms first K_n etc. are pictured in Figure 6.

Notice that we have to label the edges between the vertices in the first and the second K_n. For this we use the elements of S_5. Similarly, the elements of S_{10} are used to label the edges between the third and the fourth K_n. We are now left with the edges between the first and second K_n to the third and fourth K_n. For this we use the elements of S_{11}.

For constructing L_1 through L_4, we use a similar approach used in the proof of Theorem

3.1. We first construct a magic square of order n and replace the edge and vertex labels with the elements of the appropriate set.

We now focus on the construction of N_1 through N_4, which serve as edge labels of the edges between the first and second K_n and edges between the third and the fourth K_n. Consider a magic square of order n and replace the entries by the elements of S_5 to get N_1. We define $N_2 = N_1^T$. The entries in N_1 and N_2 thus serve as the labels of the edges between the first and second K_n. Similarly, we obtain N_3 by replacing the entries of a magic square of order n by the entries of S_{10}. We then define $N_4 = N_3^T$. The entries in N_3 and N_4 thus serve as the labels of the edges between the third and fourth K_n.

Finally, we show the construction of labels for the edges from the first and second K_n to the third and the fourth K_n. These labels are from the set S_{11} which has $4n^2$ numbers. We form these labels as four matrices M_1 through M_4 and their transposes. The matrix M_i, for $1 \leq i \leq 4$, is obtained by replacing the entries of a magic square of order n with the entries of S_{11} ranging from $(i-1)n^2 + 1$ to $i \cdot n^2$.

Finally, arrange all these intermediate labellings as shown below.

$$\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ N_3 & N_4 & N_1 & N_2 \\ M_1 & M_2 & M_1^T & M_2^T \\ M_4 & M_3 & M_4^T & M_3^T \end{bmatrix}$$

It can be deduced that the magic constant in this case is $\frac{32n^3+13n^2+n}{2}$.

As in Section 3.1, notice that a rearrangement of labels as shown below has to be done to arrive at the standard form given in Section 1.1. In the table below, the notation $M : a$ refers to the ath row of the matrix M for a positive integer a, and the notation $M : a \cdots b$ refers to the rows a through b (both inclusive) of the matrix M, for positive integers a, b, $a < b$.

$$\begin{bmatrix} L_1 : 1 & L_2 : 1 & L_3 : 1 & L_4 : 1 \\ L_1 : 2 \cdots n & N_4 : 1 \cdots n-1 & M_1^T : 1 \cdots n-1 & M_4^T : 1 \cdots n-1 \\ N_3 : 1 & N_4 : n & M_1^T : n & M_4^T : n \\ N_3 : 2 \cdots n & L_2 : 2 \cdots n & M_2^T : 1 \cdots n-1 & M_3^T : 1 \cdots n-1 \\ M_1 : 1 & M_2 : 1 & M_2^T : n & M_3^T : n \\ M_1 : 2 \cdots n & M_2 : 2 \cdots n & L_3 : 2 \cdots n & N_2 : 1 : n-1 \\ M_4 : 1 & M_3 : 1 & N_1 : 1 & N_2 : n \\ M_4 : 2 \cdots n & M_3 : 2 \cdots n & N_1 : 2 \cdots n & L_4 : 2 \cdots n \end{bmatrix}$$

□

Example 3.4. *Let us construct vertex magic total labelling for K_{12} using a vertex magic total labeling of K_3.*

Let L, shown below, be the vertex magic total labeling of K_3 obtained from using the approach of Section 2.

vertex label	04	05	06
Incident edges	03	03	02
	02	01	01

From this, the matrix L_1 is obtained by using $S_1 = \{1, 2, 3\}$ for the edge labels and $S_4 = \{13, 14, 15\}$ for the vertex labels. The resulting L_1 is shown below.

Vertex label	13	14	15
Incident edges	03	03	02
	02	01	01

Using $S_2 = \{4, 5, 6\}$, $S_3 = \{7, 8, 9\}$, $S_6 = \{22, 23, 24\}$, and $S_9 = \{34, 25, 36\}$, we obtain the matrices L_2, L_3, and L_4 respectively as described in the proof of Theorem 3.3. The resulting matrices are shown below.

$$L_2 = \begin{bmatrix} 7 & 8 & 9 \\ 6 & 6 & 5 \\ 5 & 4 & 4 \end{bmatrix} \quad L_3 = \begin{bmatrix} 34 & 35 & 36 \\ 24 & 24 & 23 \\ 23 & 22 & 22 \end{bmatrix} \quad L_4 = \begin{bmatrix} 28 & 29 & 30 \\ 27 & 27 & 26 \\ 26 & 25 & 25 \end{bmatrix}$$

To construct the matrices N_1 and N_2, consider a magic square of order three and the set $S_5 = \{10, 11, 12, 16, 17, 18, 19, 20, 21\}$. Replace the elements of the magic square by elements of S_5 to get N_1. The resulting N_1 and N_2 are shown below.

$$N_1 = \begin{bmatrix} 16 & 21 & 11 \\ 12 & 17 & 19 \\ 20 & 10 & 18 \end{bmatrix} \quad N_2 = \begin{bmatrix} 16 & 12 & 20 \\ 21 & 17 & 10 \\ 11 & 19 & 18 \end{bmatrix}$$

Similarly, using the elements of the sets $S_{10} = \{31, 32, 33, 37, 38, 39, 40, 41, 42\}$, the matrices N_3 and N_4 are obtained as below.

$$N_3 = \begin{bmatrix} 37 & 42 & 32 \\ 33 & 38 & 40 \\ 41 & 31 & 39 \end{bmatrix} \quad N_4 = \begin{bmatrix} 37 & 33 & 41 \\ 42 & 38 & 31 \\ 32 & 40 & 39 \end{bmatrix}$$

Finally, the set S_{11} has 36 consecutive elements from 43 to 78. Of these, the first nine elements are used to construct the matrices M_1, the second nine elements are used to construct M_2, the third nine elements for M_3 and the final nine elements for M_4. For purposes of brevity, we show only M_1 and M_2 in the following. The matrix M_1 is obtained as a magic square of order three with elements from 43 to 51.

$$M_1 = \begin{bmatrix} 46 & 51 & 44 \\ 45 & 47 & 49 \\ 50 & 43 & 48 \end{bmatrix} \quad M_2 = \begin{bmatrix} 55 & 60 & 63 \\ 54 & 56 & 58 \\ 59 & 52 & 57 \end{bmatrix}$$

The final matrix after combining all of these matrices is shown in Figure 7.

Vertex label	13	14	15	7	8	9	34	35	36	28	29	30
Incident Edges	3	3	2	6	6	5	24	24	23	27	27	26
	2	1	1	5	4	4	23	22	22	26	25	25
	37	42	32	37	33	41	16	21	11	16	12	20
	33	38	40	42	38	31	12	17	19	21	17	10
	41	31	39	32	40	39	20	10	18	11	19	18
	46	51	44	55	60	53	46	45	50	55	54	59
	45	47	49	54	56	58	51	47	43	60	56	52
	50	43	48	59	42	57	44	49	48	53	58	57
	73	78	71	64	69	62	73	72	77	64	63	68
	72	74	76	63	65	67	78	74	70	69	65	61
	77	70	75	68	61	66	71	76	75	62	76	66

Figure 7: A vertex magic total labeling of K_{12}. As earlier, the elements of the first row are the vertex labels and the remaining column entries are the edge labels.

The above matrix elements have to be rearranged to get the label matrix in the standard form.

3.3. Labeling for $N \equiv 0 \pmod 8$

Finally, we focus on the case where $n \equiv 0 \bmod 8$. In this case, we use mostly similar techniques as that of Theorem 3.1. Hence, we provide an outline of the proof for the following theorem.

Theorem 3.5. *There is a vertex magic total labeling for K_N, for all $N \equiv 0 \bmod 8$.*

Proof. (Outline). Notice that in this case, we can represent K_N as two $K_{N/2}$ and a $K_{N/2,N/2}$ with $N/2 \equiv 0 \bmod 4$. We then use the constructions of Theorem 3.3 and Theorem 3.1 to combine the labels of the $K_{N/2}$ with the edge labels of $K_{N/2,N/2}$ and arrive at a vertex magic total labeling of K_N with $N \equiv 0 \bmod 8$. □

4. Conclusion

In this paper, we used cohesive techniques to show that every complete graph has a vertex magic total labeling. Our techniques attempt to draw connections between graph factorization and graph labeling. It is worth to explore the stronger connections between the two.

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