Acyclic Vertex Coloring of Graphs of Maximum Degree 5[2]

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Abstract

An acyclic vertex coloring of a graph is a proper vertex coloring such that there are no bichromatic cycles. The acyclic chromatic number of G, denoted *a*(G), is the minimum number of colors required for acyclic vertex coloring of graph G = (V,E). For a family F of graphs, the acyclic chromatic number of F, denoted by *a*(F), is defined as the maximum *a*(G) over all the graphs G ∈ F. In this paper we show that *a*(F)=8 where F is the family of graphs of maximum degree 5 and give a linear time algorithm to achieve this bound.

1. Introduction

A proper coloring of the vertices of a graph G = (V,E) is an assignment of colors to the vertices so that no two neighbors get the same color. A proper coloring is said to be acyclic if the coloring does not induce any bichromatic cycles. The acyclic chromatic number of a graph G is denoted *a*(G), and is the minimum number of colors required to acyclically color the vertices of G.

The concept of acyclic coloring of a graph was introduced by Grunbaum [G73] and is further studied in the last two decades in several works, [AMR91, AB76, Bo79, BFKRS02, BKRS99, ASZ01] among others. Determination of *a*(G) is a hard problem even theoretically. For example, Kostochka [K02] proves that it is an NP-complete problem to decide for a given arbitrary graph G whether *a*(G) ≤ 3.

Given the computational difficulty involved in determining *a*(G), several authors have looked at acyclically coloring particular families of graphs. In this context, Borodin [Bo79] focuses on the family of planar graphs, the family of planar graphs with "large" girth [BKW99], 1-planar graphs [BKRS99], outer planar graphs [Sop97], d-dimensional grids [FGR03], graphs of maximum degree 3 [G73, S04], and of maximum degree 4[Bu79].

Another direction that has yielded fruits is that of using the probabilistic method and the Lovasz Local Lemma (LLL) [B91]. Using this method, it was shown by Alon et al. [AMR91] that any graph of maximum degree Δ can be acyclically colored using $O(\Delta^{4/3})$ colors, thus showing that *a*(G) ≤ $O(\Delta^{4/3})$. In the same paper, it was also shown that, as $n \to \infty$, there exist graphs with maximum degree Δ and requiring $\Omega(\Delta^{4/3}/\log \Delta^{1/3})$ colors for an acyclic coloring. The above two results are based on the probabilistic method. They further showed that a greedy algorithm exists to acyclically color any graph G with maximum degree Δ using Δ^2+1 colors. This was later improved by Albertson et al. [ACK04] to show that *a*(G) ≤ $\Delta(\Delta - 1) + 2$.

[2] This work is supported by the Department of Science and Technology, Govt. of India, New Delhi, under Project No.SR/S4/MS:406/06.

Focusing on the family of graphs with a small maximum degree Δ, it was proved by Skulrattanakulchai [S04] that $a(G) \leq 4$ for any graph of maximum degree 3. Burnstein [Bu79] showed that $a(G) \leq 5$ for any graph of degree maximum 4. The work of Skulrattanakulchai was extended by Fertin and Raspaud [FR05] to show that it is possible to acyclically vertex color a graph G of maximum degree Δ using at most $\Delta(\Delta - 1)/2$ colors. In the same paper, it was also shown that for any graph G of maximum degree 5, $a(G) \leq 9$ and there exists a linear time algorithm to acyclically color G using at most 9 colors. We improve the result of [FR05] for the case when $\Delta \leq 5$ and show that it is possible to acyclically color the vertices of a graph G with maximum degree at most 5 using only 8 colors. Below, we first introduce the notation that is used in the rest of the paper.

1.1 Notation

For a positive integer k, $[k]$ refers to the set of positive integers $\{1, 2, \ldots, k\}$. We stick to standard graph theoretic notation (cf. [W04]) for terms not defined here. We use notation from [S04], and repeat it for sake of clarity. We start with the following definition.

Definition 1.1 Let $W \subseteq V(G)$. The neighborhood of W, denoted N(W), is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in W. A neighbor of W is a vertex in N(W), $N(v)$ stands for $N(\{v\})$.

Definition 1.2 A partial coloring is an assignment of colors to a subset of V(G) such that the colored vertices induce a graph with an acyclic coloring.

Suppose G has a partial coloring. Let α, β be any two colors. An *alternating* α, β– path is a path in G with each vertex colored either α or β. An alternating path is an alternating α, β– path for some colors α, β. A path is odd or even according to the parity of number of edges it contains. Let v be an uncolored vertex. A color $\alpha \in [8]$ is available for v if no neighbor of v is colored α. A color $\alpha \in [8]$ is feasible for v if assigning color α to v still results in a partial coloring. (Thus feasibility implies availability but not the other way around). Let C_v be a cycle in G containing vertex v. A cycle C_v is α, β– dangerous if $C_v - v$ is an even α, β--alternating path. A cycle C_v is dangerous if it is α, β– dangerous for some colors α, β.

Definition 1.3 A vertex v is called a single vertex if all its colored neighbors receive distinct colors.

Notice that the above definition also treats a vertex v with some uncolored neighbors but the colored neighbors having distinct colors as a single vertex. The notion of a single vertex is useful because recoloring is easy at single vertices.

1.2 Our Results

In this paper, we show that any graph G with maximum degree bounded by 5 can be acyclically colored using 8 colors. We show this result by extending a partial coloring by one vertex at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of v, in all cases. Specifically, we show the following lemmas which result in Theorem 1.7.

Lemma 1.4 Let π be any partial coloring of G using colors in [8] and let v be any uncolored vertex. If v has less than 4 colored neighbors, then there exists a color $\alpha \in [8]$ feasible for v. Moreover, such a color α can be found in O(1) time.

Lemma 1.5 Let α be any partial coloring of G using colors in [8] and let v be any uncolored vertex. If v has exactly four colored neighbors, then there exists a partial coloring π_1 of G using colors in [8] and a color $\alpha \in$ [8] satisfying the following conditions.

- π_1 has the same domain as π.
- $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$.
- α is feasible for v under π_1.

Moreover, both π_1 and α can be found in O(1) time.

Lemma 1.6 Let π be any partial coloring of G using colors in [8] and let v be any uncolored vertex. If v has five colored neighbors, then there exists a partial coloring π_1 of G using colors in [8] and a color $\alpha \in$ [8] satisfying the following conditions.

- π_1 has the same domain as π.
- $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ or $x \in N(N(v))$.
- α is feasible for v under π_1.

Moreover, both π_1 and α can be found in O(1) time.

Theorem 1.7 The vertices of any graph G of degree at most 5 can be acyclically colored using eight colors in O(n) time, where n is the number of vertices.

The rest of the paper is organized as follows. In Section 2, we prove Lemma 1.4. In Section 3 and 4, we prove lemma 1.5 and 1.6 respectively. The paper ends with some concluding remarks in Section 5.

2. Proof of Lemma 1.4

Vertex v has at most 5 neighbors. We may assume without loss of generality that all 5 neighbors u, w, x, y, z exist. We then split the proof into 3 cases.

1. At most 1 neighbor of v is colored: Then v has at least 7 available colors and all of them are obviously feasible for v.
2. v has exactly 2 colored neighbors: Let they are w and x. Suppose $\pi(w) \neq \pi(x)$. As no neighbors of vertex v have same color remaining 6 available colors are feasible to vertex v. Suppose $\pi(w) = \pi(x)$. Since two colors are same, the remaining seven colors are available to v. Out of the seven available colors at most four colors may involve in four dangerous cycles. Hence the other three colors are feasible to v.
3. v has 3 colored neighbors: We further subdivide this case into three cases.
    a. No neighbor of v has the same color: In this case, v has 5 available colors, and all of them are feasible for v.

b. Two neighbors of v have the same color but third neighbor has a different color from these two: Assume without loss of generality that $\pi(w) = \pi(x) = 1$ and $\pi(y) = 2$. Thus colors 3, 4, 5, 6, 7, and 8 are available for v. Any dangerous cycle involving v is 1,β--dangerous for some β and it involves the path $\{w, v, x\}$. As $\Delta = 5$, no more than 4 colors can be the color β in any 1, β--dangerous cycle involving v, i.e, .there are at least 2 colors are feasible for v.

c. All the three neighbors of v have the same color: Thus 7 colors are available to v. Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y)=1$. Each colored neighbor of v, i.e., any vertex in $\{w, x, y, z\}$, can participate in at most three dangerous C_v cycles and each C_v cycle contains two neighbors of v. Hence there are at most 6 dangerous C_v cycles. Since v has 7 colors available but at most 6 of them are not feasible, at least one of them is feasible.

In all cases above, a feasible color for v can be found in O(1) time.

3 Proof of Lemma 1.5

In this section we deal the case where v has exactly 4 colored neighbors. We deal the proof in several cases depending on the colors of the neighbors of v.

1. Case 1 – If no neighbors of v have the same color, then v has 4 available colors, and all of them are feasible for v.
2. Case 2 – If two neighbors have same color and two are different: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, and $\pi(y) = 2$, and $\pi(z) = 3$. Any dangerous cycle involving v is 1,β– dangerous for some β and it involves the path $\{w, v, x\}$. As $\Delta \leq 5$, no more than 4 colors can be the color β in any 1,β– dangerous cycle involving v. Since v has 5 available colors and at most 4 colors are not feasible, at least one of them is feasible.
3. Case 3 – If three neighbors have same color and one is different: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$ and $\pi(z) = 2$ then v has 6 available colors. Any dangerous cycle involving v is 1,β– dangerous for some β and it involves the path $\{w, v, x\}$ or $\{x, v, y\}$ or $\{w, v, y\}$. Any vertex in $\{w, x, y\}$ can participate in at most four dangerous C_v cycles each C_v cycle contains two neighbors of v. Assume they are 1,3–, 1,4–, 1,5–, 1,6–, 1,7– and 1,8– dangerous, for otherwise there is obviously an available color that is feasible. Assume without loss of generality that edge vw is involved in 1,3–, 1,4–, 1,5– and 1,6– dangerous cycles. This implies that the neighbors of w colored with colors 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertex t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 above.
4. Case 4 – If all the four neighbors have same color: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$. Any vertex in $\{w, x, y, z\}$ can participate in at most four dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence there are at most 8 dangerous C_v cycles. Thus, none of the available color can be feasible for v. We now deal with two cases.
    a. If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 3 above.
    b. If none of $\{w, x, y, z\}$ is a single vertex: In this case, any vertex in $\{w, x, y, z\}$ has at most 3 different colored vertices. Thus, any neighbor of v can participate in at most 3 dangerous C_v

cycles, and each C_v cycle contains two neighbors of *v*. Hence, there are at most 6 dangerous C_v cycles. Since *v* has 7 available colors but at most 6 of them are not feasible, at least one of them is feasible.

5. Case 5 – If exactly two neighbors have the same color and another two neighbors have same but different color: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, and $\pi(y) = \pi(z) = 2$. There are at most four 1,β– dangerous C_v cycles for some β involving the path {*w, v, x*} and at most four 2, β– dangerous C_v cycles for some β involving the path {*y, v, z*}. Hence, presently, no color is feasible for *v*. To find a feasible color for *v*, we make the following case distinction.
   a. If any of the neighbors of *v* is a single vertex: Assume without loss of generality that *w* has neighbors colored 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices *t*. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 2 of this section.
   b. If none of {*w, x, y, z*} is a single vertex: If no available color is feasible for *v* then there must be three 1,β– dangerous C_v cycles and three 2,β– dangerous C_v cycles. Assume without loss of generality they are 1,3–, 1,4–, 1,5–, and 2,6–, 2,7–, 2,8– dangerous cycles. This implies that three neighbors of *w* are colored 3, 4, 5 and another neighbor should be colored any of {3, 4, 5}. Let it be 3. Let the like neighbors of *w* be w_1, w_2, i.e., w_1 and w_2 are colored with color 3. The available colors for recoloring *w* is any of {2, 6, 7, 8}. Suppose we define π_1 by setting $\pi_1(w) = k$ for $k \in \{2, 6, 7, 8\}$ and setting, $\pi_1(t) = \pi(t)$ for all other colored vertices *t*. For π_1 not to be partial coloring, both the alike neighbors of *w* should contain {2, 6, 7, 8} as the colors of neighbors. Since then there is no possibility of 1,3– dangerous cycles, the color 3 is feasible to *v*. Thus, *k* can be chosen as one of {2, 6, 7, 8} that is free at the alike neighbors of *w*. Moreover, under π_1 this case converts to Case 3 if k = 2, or Case 2 if $k \in \{6, 7, 8\}$.

4 Proof of Lemma 1.6

Suppose *v* has 5 colored neighbors. We deal the proof of Lemma 1.6 by breaking it into several cases.

1. Case 1 – If no neighbors of *v* have the same color: Then *v* has 3 available colors, and all of them are feasible for *v*.
2. Case 2 – If two neighbors are same colored and three are different colored: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = 2$, $\pi(z) = 3$, and $\pi(u) = 4$. Any dangerous cycle involving *v* is 1, β– dangerous for some β and it involves the path {*w, v, x*}. As $\Delta = 5$, no more than 4 colors can be the color β in any 1,β– dangerous cycle involving *v*. Assume they are 1,5–, 1,6–, 1,7– and 1,8– dangerous cycles. Otherwise there is obviously an available color that is feasible. This implies that neighbors of *w* are colored 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 2$, $\pi_1(x) = 2$, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices *t*. Then, π_1 is also a partial coloring. Moreover, under π_1, the color 1 is feasible for *v*. Thus, finding the asserted π_1 can be done in O(1) time.
3. Case 3 – If three neighbors are colored with the same color and two neighbors are differently colored: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, $\pi(z) = 2$, and $\pi(u) = 3$. Any of {*w, x, y*} can participate in at most 4 dangerous cycles and each C_v cycle contains two neighbors of *v*. Hence there are at most 6 dangerous C_v cycles. Thus, all the five available colors may not be feasible for *v*. So, we differentiate between two cases.

a. If any of $\{w, x, y\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 of Section 4.

b. If none of $\{w, x, y\}$ is a single vertex: Any vertex in $\{w, x, y, z\}$ has at most 3 different vertices. Thus, any neighbor of v can participate in at most 3 dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence there are at most 4 dangerous C_v cycles. Since v has 5 available colors but at most 4 of them are not feasible, at least one of them is feasible. Such a feasible color can be found in O(1) time.

4. Case 4 – When two neighbors are colored with the same color and two other are colored with the same (but different from the above) color, and another one is colored differently: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = \pi(z) = 2$, and $\pi(u)=3$. There are at most four $1,\beta$– dangerous C_v cycles for some β involving the path $\{w, v, x\}$ and at most four $2, \beta$– dangerous C_v cycles for some β and it involves the path $\{y, v, z\}$. Since v has 5 available colors and all of them can be involved in dangerous C_v cycles, no color is feasible for v. So we proceed by making the following case distinction.

   a. If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of Section 4.

   b. If none of $\{w, x, y, z\}$ is a single vertex: For no available color to be feasible for v there must be at least three $1,\beta$– dangerous C_v cycles and two $2, \beta$– dangerous C_v cycles. Assume without loss of generality they are 1,4–, 1,5–, 1,6–, and 2,7–, 2,8– dangerous C_v cycles. This implies that three neighbors of w and x are colored with colors 4, 5, and 6, and another neighbor should be colored with any of $\{4, 5, 6\}$. Let it be 4. The available colors for recoloring w are $\{2, 3, 7, 8\}$. Let the like neighbors of w be w_1, w_2, i.e., w_1 and w_2 are colored with color 4. When any of $\{2, 7, 8\}$ is missing in the neighborhood of any of w_1 or w_2, we define π_1 by setting $\pi_1(w) = k$ where k is one among the missing colors, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of Section 4 if $k = 2$, or Case 2 of Section 4 if $k \in \{7, 8\}$. Otherwise w_1 and w_2 have neighbors colored with colors $\{2, 7, 8\}$. Let the other neighbor of w_1 and w_2 be colored a, and b respectively.

   Consider the neighbor of w_1 colored a. (If that neighbor is not colored yet, we take that $a = -1$). When a is different from $\{2, 7, 8\}$ then define π_1 as follows:

$$\pi_1(w) = 7 \text{ or } 8$$
$$\pi_1(w_1) = \begin{cases} 1 & \text{if } a \neq 1 \\ 3 & \text{if } a = 1 \end{cases}$$
$$\pi_1(t) = \pi(t) \text{ for all other colored vertices } t.$$

Otherwise, the color a is one among {2, 7, 8}. In that case, consider the neighbor of w_1 colored b. When b is different from {2, 7, 8} then define π_1 as earlier.

When b is also one among {2, 7, 8} then there is no possibility of 1,4– dangerous cycles. The color 4 is free from 1,4– dangerous C_v cycles. If there is no possibility of 2,4– dangerous cycles, then the color 4 is feasible for v. Otherwise both of the neighbors of y and z should be colored 4, i.e., colored neighborhood of y and z contain {4, 7, 8} only.

In this case, similarly consider vertex x, let the like neighbors of x be x_1 and x_2. When any of {2, 7, 8} is missing in the neighborhood of any of x_1 or x_2, we define π_1 by setting $\pi_1(x) = k$ where k is one among the missing colors, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of Section 4 if $k = 2$, or Case 2 of Section 4 if $k \in \{7, 8\}$. Else if 3 is missing in the neighborhood of any of x_1 or x_2 define π_1 by setting $\pi_1(x) = 3$, $\pi_1(w) = 3$, $\pi_1(t) = \pi_1(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 color 1 is feasible for v as there is no possibility of 2,1– and 3,1– dangerous cycles under π_1. For an illustration, please refer Figure 1.

Otherwise x_1 and x_2 have neighbors colored with colors {2, 3, 7, 8}.Then define $\pi_1$by setting $\pi_1(x) = 7$ or 8 , $\pi_1 (x_1) = 1$ and $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 of Section 4.

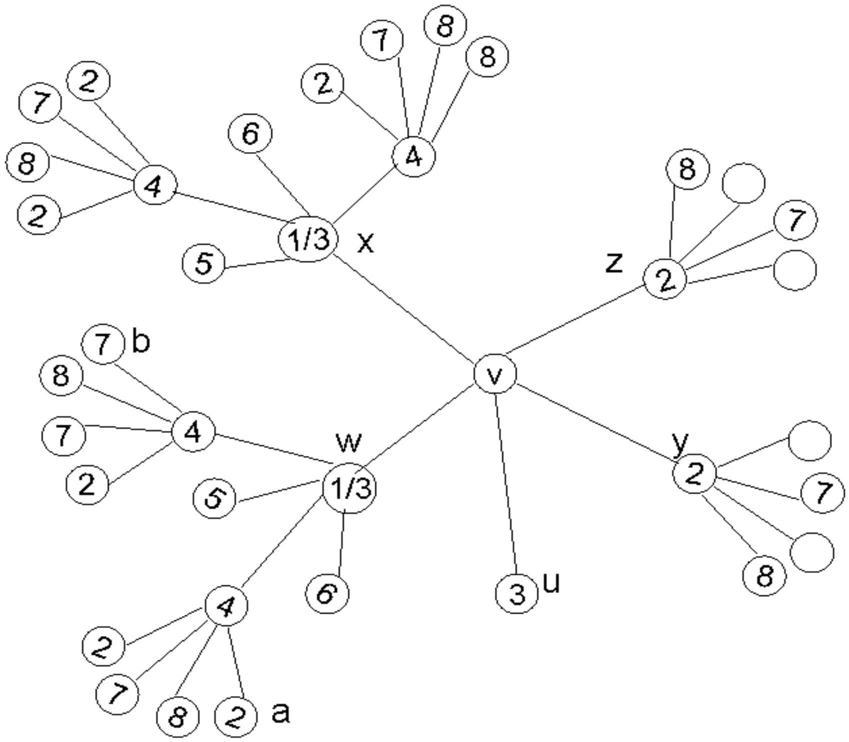

Figure 1: The figure illustrates the recoloring of vertices *w* and *x* to color 3, the colors *a* and *b* are 2 and 7 respectively. The neighbors of *z* and *u* are colored only among colors 4, 7, and 8 as shown. After this recoloring, color 1 is thus made feasible for *v*. The numbers inside the vertices denote the colors of the vertices.

5. Case 5 – If four neighbors are colored with the same color and one neighbor is differently colored: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$, and $\pi(u)=2$. Any of {*w, x, y, z*} can participate in at most 4 dangerous cycles and each such C_v cycle contains two neighbors of *v*. Hence there are at most 8 dangerous C_v cycles. Thus all the six available colors may not be feasible for *v*. We proceed in the following way by making a case distinction.
   a. If any of {*w, x, y, z*} is a single vertex: Assume without loss of generality that *w* has neighbors colored with colors 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices *t*. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of Section 4.
   b. If none of {*w, x, y, z*} is a single vertex: Any vertex in {*w, x, y, z*} has at most 3 differently colored vertices. Thus any neighbor of *v* can participate in at most 3 dangerous C_v cycles, and each such C_v cycle contains two neighbors of *v*. Hence there are at most 6 dangerous C_v cycles. Assume without loss of generality they are 1,3–, 1,4–, 1,5–, 1,6–, 1,7–, and 1,8–dangerous C_v cycles. (Otherwise, there is obviously an available color that is feasible for *v*.) This implies that every available color is neighbor of exactly two of {*w, x, y, z*}. Assume

without loss of generality w has neighbors colored 3, 3, 4 and 5. Let the like neighbors of w be w_1 and w_2, i.e., w_1 and w_2 are colored with color 3.

When any of the colors in {6, 7, 8} is missing in the neighborhood of any of w_1 or w_2, then define π_1 by setting $\pi_1(w) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of Section 4. Otherwise, w_1 and w_2 have neighbors colored with colors in {6, 7, 8}. Let the other neighbor of w_1 and w_2 be colored a and b respectively. (If those nodes are not colored, we take them to be colored with -1).

Consider the neighbor of w_1 colored a. When $a \notin \{6, 7, 8\}$, define π_1 as follows.

$$\pi_1(w) = 6 \text{ or } 7 \text{ or } 8$$

$$\pi_1(w_1) = \begin{cases} 2 \text{ if } a \neq 2 \\ 1 \text{ if } a = 2, \text{ and} \end{cases}$$

$$\pi_1(t) = \pi(t) \text{ for all other colored vertices } t.$$

Moreover, under π_1 this converts to Case 3 of Section 4. When $a \in \{6, 7, 8\}$, then consider the neighbor of w_2 colored b. When $b \notin \{6, 7, 8\}$, then define π_1 similarly as above. When $b \in \{6, 7, 8\}$, then notice that there is no possibility of 1,3– dangerous cycles. Thus, the color 3 is feasible for v.

6. Case 6 – If three neighbors are colored with the same color and the two remaining neighbors are colored with the same, but different from above color: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$ and $\pi(z) = \pi(u) = 2$. Any of {w, x, y} can participate in at most four 1,β– dangerous cycles and each such C_v cycle contains two neighbors of v. Hence, there are at most six 1,β– dangerous C_v cycles. Similarly, there are at most four 2,β– dangerous C_v cycles. Hence, none of the available colors can be feasible for v. During the recoloring process, we proceed as follows.
   a. If any of the {*w, x, y, z, u*} is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 4 of Section 4.
   b. If none of {*w, x, y, z, u*} is a single vertex: In this case, any vertex in {*w, x, y*} has at most 3 differently colored neighbors. Thus, any of {*w, x, y*} can participate in at most 3 dangerous C_v cycles, and each such C_v cycle contains two neighbors of v. Hence, there are at most 4 1,β– dangerous C_v cycles. By a similar reasoning, there can be at most three 2,β– dangerous cycles.
      - If any of {*z, u*} neighborhood colored with only three colors: Assume without loss of generality that they are 3, 4, and 5, i.e, z has three neighbors colored with {3, 4, 5} and remaining neighbor should be colored any of {3, 4, 5}. Let it be colored with 3. Let the like neighbors of z be z_1 and z_2, i.e., both z_1 and z_2 are colored with

3. When any of {1, 6, 7, 8} is missing in the neighborhood of any of z_1 or z_2, then define π_1 by setting $\pi_1(z) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$, for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 3 of Section 4 if $k \in \{6, 7, 8\}$, or to Case 5 of Section 4, if $k = 1$.

Otherwise, both z_1 and z_2 have neighbors colored with colors in {1, 6, 7, 8}. Then we recolor as follows. Define π_1 by setting $\pi_1(z) = 6$, $\pi_1(z_1) = 2$, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Moreover, under π_1, this case converts to Case 3 of Section 4.

- If both of $\{z, u\}$ neighborhood colored with less than three colors there should be exactly two 2,β– dangerous cycles and four 1,β– dangerous cycles. Otherwise there is obviously an available color that is feasible. Assume without loss of generality they are 1,3–, 1,4–, 1,5–, 1,6–, and 2,7–, 2,8– dangerous C_v cycles, i.e., z and u neighborhood exactly colored with {7, 8} only. Now let us consider the neighborhood of vertices w, x, and y. If all of $\{w, x, y\}$ have only 2 differently colored neighbors, then every one of $\{w, x, y\}$ can participate in at most 2 dangerous C_v cycles and each such C_v cycle contains two neighbors of v. Hence there can be at most three 1,β– dangerous C_v cycles. Thus, at most three available colors can participate in dangerous cycles and the other two available colors cannot participate in dangerous cycles. Thus, it must be the case that two neighbors exist with three differently colored so that no available color is feasible for v. Let the two such neighbors be w and x. Assume without loss of generality that w has neighbors colored with colors 3, 3, 4, and 5. Let the like neighbors of w be w_1 and w_2, i.e., both w_1 and w_2 are colored with color 3.

  When any of the colors {6, 7, 8} is missing in the neighborhood of any of w_1 or w_2, we define π_1 by setting $\pi_1(w) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 5 of Section 4.

  Otherwise, w_1 and w_2 have neighbors colored with colors {6, 7, 8}. Let the other neighbor of w_1 and w_2 be colored a, and b respectively. (If those nodes are not colored yet, we take the colors to be -1). Consider the color a. When a is different from {6, 7, 8}, define π_1 as follows:

$$\pi_1(w) = 6 \text{ or } 7 \text{ or } 8$$

$$\pi_1(w_1) = \begin{cases} 2 \text{ if } a \neq 2 \\ 1 \text{ if } a = 2, \text{ and} \end{cases}$$

$$\pi_1(t) = \pi(t) \text{ for all other colored vertices } t.$$

Moreover, under π_1 this converts to Case 4 of Section 4.

When a is one among {6, 7, 8} then consider b. When b is different from {6, 7, 8} then define π_1 as same above. When b is one among {6, 7, 8}, we define π_1 by setting $\pi_1(w) = 2$ and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, under π_1, w cannot participate in any dangerous cycles and the four 1,β– dangerous cycles should be formed through x-v-y path only. This implies that x and y are single vertices, which is not true according to our assumption earlier. Hence, under π_1 there is an available color that is feasible for v, which can be found by looking at the colored neighbors of x and y.

7. Case 7 – When all the five neighbors are colored with the same color: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = \pi(u) = 1$. Any of $\{w, x, y, z, u\}$ can participate in at most 4 dangerous C_v cycles and each C_v cycle contains two neighbors of v. Hence there are at most 10 dangerous C_v cycles. Thus, all the seven available colors may not be feasible for v. Hence, we proceed as follows.
   a. If any of $\{w, x, y, z, u\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 2, 3, 4, and 5. Define π_1 by setting $\pi_1(w) = 6$ and setting $\pi_1(t) = \pi_1(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 5 of Section 4.
   b. If none of $\{w, x, y, z, u\}$ is single vertex: Any vertex in $\{w, x, y, z, u\}$ has at most 3 different vertices. Assume every neighbor of v has 2 different neighbors. Then every neighbor of v can participate in at most 2 dangerous C_v cycles and each such C_v cycle contains two neighbors of v. Hence there are at most five dangerous C_v cycles. Thus, at most five available colors can participate in dangerous cycles and the other two available colors cannot participate in dangerous cycles. So we can find a feasible color for v.

Otherwise, there exist two neighbors with three different children so that no available color is feasible. Let the two neighbors be w and x. Assume without loss of generality that w has neighbors colored 2, 2, 3, and 4. Let the like neighbors of w be w_1 and w_2, i.e., w_1 and w_2 are colored with color 2. When any of {5, 6, 7, 8} is missing in the neighborhood of any of w_1 or w_2, then define π_1 by setting $\pi_1(w) = k$, where k is one among missing colors, and setting $\pi_1(t) = \pi(t)$ for all other colored vertices t. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 5 of Section 4. Otherwise, w_1 and w_2 have neighbors colored {5, 6, 7, 8}. Then, define π_1 by setting $\pi_1(w) = 5$, $\pi_1(w_1) = 1$ and setting $\pi_1(\mathrm{t}) = \pi(t)$ for all other colored vertices t. Moreover, under π_1, this case converts to Case 5 of Section 4.

5 Conclusions

In this paper, we have presented a polynomial time algorithm to acyclically color the vertices of graphs whose maximum degree is bounded by 5. The algorithm improves the state-of-the-art by 1 color, by a careful consideration of the various cases. It is also possible to extend our result to acyclically color the vertices graphs of degree at most 6. Our preliminary results indicate that in this case, 12 colors suffice.

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