Acyclic Edge Coloring Algorithms for $K_{p(q-1)}$ and $K_{(p-1)(q-1),(p-1)(q-1)}$.*

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Abstract

Let $p, q,$ and r denote prime numbers. In this paper, simple linear time algorithms to acyclically color the edges of complete graphs of order $p(q-1)$, a complete graphs of order $p(q-1)(r-1)$, and a complete bipartite graphs of order $(p-1)(q-1)$ are presented. The number of colors used by our algorithms improve the state-of-the-art and is close to the optimal value for small values of p. All the above algorithms are based on a simple algorithm for general graphs of order n that uses $2n-3$ colors. We also present a variation of the simple algorithm that uses p colors for a complete graph of order p, which resembles the algorithm of Alon et al.[2].

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1 Introduction

An edge coloring of a graph G is proper if no two incident edges have the same color. It is acyclic if it is proper and there is no cycle in the subgraph induced by the edges of any two of the colors. The acyclic edge chromatic number of a graph G, denoted by $a\,'(G)$, is the least number of colors in an acyclic edge coloring of G.

Acyclic colorings were introduced by Grunbaum [11]. The subject is further studied by Albertson and Berman [1], Borodin [6, 7, 8, 9], Alon [2, 3, 4], amongst others. Throughout the paper p, q, and r denote prime numbers greater than 3, unless otherwise stated.

Determination of the edge chromatic number, $a\,'(G)$, of a graph G is a hard problem both from theoretical and algorithmic point of view. For example, Alon and Zaks prove in [4] that it is an NP-complete problem to decide for a given arbitrary graph G whether $a\,'(G) \leq 3$. However, the following upper bounds have been obtained. As a corollary to a result on acyclic vertex coloring Alon et al. [2] show that the edges of any graph G with maximum degree Δ can be acyclically colored using at most 64Δ colors. Molloy and Reed [15] improved on the constant from 64 to 16. Alon et al. [3] claim that the constant 16 can be further improved. They further conjecture that $a\,'(G) \leq \Delta(G) + 2$ for all graphs G and prove that there exists a constant c such that $a\,'(G) \leq \Delta(G) + 2$ for any graph G whose girth is at least $c\Delta(G)\log\Delta(G)$. In addition, they showed that $a\,'(G) \leq \Delta(G) + 2$ for almost all Δ-regular graphs. Muthu et al. [16] show that the acyclic chromatic index $a\,'(G) \leq 6\Delta$ for all graphs G with girth at least 9. The same argument was extended to achieve a bound of 4.52Δ with girth being at least 220. Nesetril and Wormald [17] have obtained a bound of $a\,'(G) \leq \Delta + 1$ for random Δ-regular (Δ-fixed) graph. All these proofs are based on probabilistic methods and are not constructive.

The difficulty in determining the value of $a\,'(G)$ for a complete graph G = K_n may be estimated from a closely related conjecture called the perfect 1-factorization conjecture [3, 18, 21, 22]. This conjecture of Kotzig [13] and others is still open except for certain values of n [3, 18]. It says that for any $n \geq 2$, K_{2n} can be decomposed into $2n-1$ perfect matchings such that the union of any two matchings forms a Hamiltonian cycle of K_{2n}. If such a decomposition of K_{2n+2} exists, then by coloring every perfect matching using a different color and removing one vertex at the end of coloring, we obtain an acyclic edge coloring of K_{2n+1} with $2n+1 = \Delta(K_{2n+1}) + 1$ colors. By removing another vertex from the colored K_{2n+1}, we obtain an acyclic edge coloring of K_{2n} with $2n+1 = \Delta(K_{2n}) + 2$ colors, which is best possible for K_{2n} [3]. Thus, if the perfect 1-factorization conjecture is true then $a\,'(K_{2n}) = a\,'(K_{2n+1}) = 2n+1$ for every n. A decomposition of K_{2n+1} into $2n+1$ matchings each having n edges such that the union of any two matchings forms a Hamiltonian path of K_{2n+1} is called a perfect near 1-factorization. As discussed above, if K_{2n+2} has a perfect 1-factorization then K_{2n+1} has a perfect near 1-factorization, which in turn implies that $a\,'(K_{2n+1}) = 2n+1$. If K_{2n+1} has an acyclic edge coloring with $2n+1$ colors then it can be shown that this coloring corresponds to a perfect near 1-factorization of K_{2n+1} which implies that K_{2n+2} has a perfect 1-factorization [3].

There has been very little algorithmic study of acyclic edge coloring except for the following works. Molloy and Reed [15] provided a general framework that can be used to develop algorithms for applications of the Lovasz Local Lemma. Customization of this general framework to acyclic coloring lead to a polynomial time algorithm to construct β-frugal coloring [15]. They also remarked that this method can be applied to find an acyclic edge coloring of a graph with maximum degree Δ using at most 20Δ colors.

Subramanian proposed a simple polynomial time greedy heuristic that uses at most $5\Delta(\log\Delta + 2)$ colors to find an acyclic edge coloring of an arbitrary graph [20]. Burnstein [10] showed that acyclic chromatic number $a(G)$ of G is at most 5 if $\Delta(G) = 4$. Since any acyclic vertex coloring of the line graph $L(G)$ is an acyclic edge coloring of G and vice-versa, this implies that $a'(G) = a(L(G)) \leq 5$ if $\Delta(G)$=3. Alon et al. [3] claim that they have another proof for this case, which yields a polynomial algorithm to acyclically edge color a sub-cubic graph using 5 colors. San Skulrattanakulchai [19] presented a first linear time algorithm to acyclically edge color a sub-cubic graph using at most 5 colors.

In view of the discussion relating acyclic edge coloring to perfect 1-factorization conjecture, it may be inferred that finding the exact values of $a'(K_n)$ for every n seems hard. However, Alon et al. [2] designed an algorithm that can acyclically edge color K_p. Through this work, they constructively showed that $a'(K_p) = p$. This corresponds to the known construction proving that K_p has a perfect near 1-factorization [21]. They also gave another algorithm that can acyclically edge color a complete bipartite graph $K_{p-1,p-1}$ and uses p colors. They further showed that $a'(K_{p-1,p-1}) = p$. The algorithm that acyclically edge colors a complete graph on prime number of vertices may be used to acyclically edge color a complete graph on an arbitrary number, n, of vertices by taking p to be the least prime greater than or equal to n. By the known results about the distribution of primes, it may be inferred that the resulting coloring requires $n+O(n^{2/3})$ colors [2]. Similarly, the number of colors required for a complete bipartite graph $K_{n,n}$, n arbitrary, using the above mentioned algorithm of Alon et al. is $n + O(n^{2/3})$ [2].

In this paper a simple linear time algorithm to acyclically edge color K_n, for any n, using at most $2n - 3$ colors is presented. This algorithm sets the tone for the rest of the algorithms that follow. First, this simple algorithm is modified to arrive at an acyclic edge coloring algorithm that uses p colors to color K_p. The resulting algorithm resembles that of Alon et al. for complete graphs with prime number of vertices. Also proposed is a linear time algorithm to acyclically edge color a complete graph on $n = p(q - 1)$ vertices using pq colors. This would improve the state of the art for a large class of values of p and q. Observe that pq being more by $(p - 1)$ from $p(q - 1) + 1$ = $\Delta(K_{p(q-1)}) + 2$, which is the optimal value[3], the number of colors that this algorithm uses is close to optimum for small values of p. The latter technique is further extended to graphs of order $p(q - 1)(r - 1)$. The extended algorithm uses pqr colors and improves on the number of colors used for certain values of p, q, and r. We also consider acyclic edge coloring of complete bipartite graphs, $K_{n,n}$ where $n = (p - 1)(q - 1)$. Our algorithm in this case uses pq colors, which can be seen to be better for some choices of p and q compared to the smallest prime greater than n. All these algorithms are simple and provide an explicit color assignment to the edges of the concerned graph.

1.1 A Note on Notation

In the rest of the paper, we use lower case a, b, c, d for colors, lower case u, v, w for vertices, and lowecase i, j, k for indices into appropriate sets. Similarly, lower case p, q, r are taken to be prime numbers. We also follow standard graph-theoretic notation whereever needed (cf. [23]).

1.2 Organization of the Paper

The rest of the paper is organized as follows. Section 2 discusses the proposed simple linear time algorithm for complete graphs of arbitrary order and its variation to complete graphs of prime order. In Section 3, an algorithm for complete graphs of order $p(q - 1)$ is presented. Its extension to graphs of order $p(q - 1)(r - 1)$ is presented in Section 4. The paper ends with some concluding remarks.

2 A Simple Algorithm and Its Variation

A simple algorithm that is designed for a complete graph of order n and uses $2n - 3$ colors is proposed here. It may be observed that the algorithm is applicable to any simple graph of finite order.

Simple Algorithm Color the edge (u, v) with the color numbered $u + v - 2$, $u, v \in \{1, 2, \cdots, n\}$.

Observation 2.1 *The algorithm uses at most $2n - 3$ colors, because the largest value either i or j can take is n and the graph is simple.*

Claim 2.2 *Coloring provided by the above algorithm is a proper edge coloring.*

Proof. Let $u \in \{1, 2, \cdots, n\}$ be an arbitrary vertex. Fix an u. Since the color of other vertex v of every edge incident at u is different from that of another edge incident at u, it follows that $u + v - 2$ (color given to the edge (u, v)) is different for every edge incident at u. □

Claim 2.3 *The union of any two color classes will not have an even length cycle.*

Proof. Consider a bichromatic path of edges colored with colors a and b starting with a vertex u. Assume an edge e colored a is incident at vertex u. Then, by the coloring principle of the algorithm, the other vertex of this edge e is $(a + 2 - u)$. Now consider the edge f incident at vertex $(a + 2 - u)$ and is colored b. Again, by the coloring principle, the other vertex of the edge f is $b - a + u$. The next vertex in this bichromatic path is $2a + 2 - b - u$. Similarly, the next vertex is $2b - 2a + u$ and so on. In general, the vertices on this bichromatic path are $\ell a + 2 - (\ell - 1)b - u$ and $\ell b - \ell a + u$, ℓ is a positive integer corresponding to the length of the path. For this bichromatic path to be a bichromatic cycle of even length, we need $\ell b - \ell a + u$ to be equal to ℓ. This implies that a must be equal to b. But a and b are two different colors – a contradiction. Hence the claim. □

Remark 2.4 *Since bichromatic odd cycles can not exist in any proper edge coloring, it follows that the coloring provided by the algorithm is an acyclic edge coloring.*

2.1 A Variation to the Simple Algorithm

The simple algorithm presented earlier may be modified to improve on the number of colors required to color a graph with prime number of vertices. The resulting algorithm requires only p colors to color a graph with p vertices. Incidentally, this algorithm resembles that of Alon et al. [2] for acyclically coloring a K_p.

Algorithm Color the edge (u, v) with the color $(u + v - 2) \bmod p$, where $u, v \in \{0, 1, \cdots, p - 1\}$.

Observation 2.5 *The algorithm may be seen to use p colors.*

Claim 2.6 *The coloring provided by the above algorithm is a proper edge coloring.*

Proof. Let $u \in \{0, 1, 2, \cdots, p - 1\}$ be an arbitrary vertex. Suppose two edges incident at u have the same color. That is, $(u + v - 2) \bmod p = (u + w - 2) \bmod p$ for some v and w. This implies $v \equiv w \bmod p$. That is, p divides $v - w$. But $0 \leq v - w \leq p - 1$ and p is a prime number. This implies that $v = w$. That is, both the edges incident at u that have the same color must be one and the same. Hence the claim. □

Claim 2.7 *The union of any two color classes will not have an even length cycle (i.e. cycle with even number of edges).*

Proof. Let a and $b \in \{0, 1, 2, \cdots, p-1\}$ be two arbitrary colors. Consider a bichromatic path of edges colored with colors a and b starting with a vertex u. Assume that an edge e colored a is incident at vertex u. Then, by the coloring principle, the other vertex of the edge e is $(a + 2 - u) \bmod p$. Similarly, the other vertices on this path are $(b - a + u) \bmod p$, $(2a + 2 - b - u) \bmod p$, $(2b - 2a + u) \bmod p$, etc. In general, an edge with color a is incident on the vertices of the form $((\ell - 1)b - (\ell - 1)a + u) \bmod p$ and $(\ell a + 2 - (\ell - 1)b - u) \bmod p$. Similarly, an edge with color b is incident on the vertices of the form $(\ell a - (\ell - 1)b + 2 - u) \bmod p$ and $(\ell b - \ell a + u) \bmod p$. For this bichromatic path to be a bichromatic cycle of even length we need $(\ell b - \ell a + u) \bmod p = u$. This implies $\ell(b - a) = kp$, for some integer k. Since p is a prime number and $0 \leq a, b \leq p - 1$, p cannot divide $(b - a)$. So, p must divide ℓ. But the value of ℓ is at most $p - 1$ and hence we arrive at a contradiction. □

3 Algorithm for Complete Graphs with $p(q-1)$ Vertices

In this section we present an algorithm to acyclically edge color a complete graph K_n, where $n = p(q-1)$ using pq colors. The main idea of this algorithm is to treat $K_{p(q-1)}$ as a complete (multi) graph on p vertices where each vertex corresponds to a complete graph on $(q-1)$ vertices. Now this complete graph on p vertices can be colored using p colors. Similarly K_{q-1} can be colored using at most q colors. Treating each multiedge in the K_p as a $K_{q-1,q-1}$, this can be colored using at most q colors. This now can be used to acyclically edge color K_n using pq colors. It may be observed that this improves the result of Alon et al. [2] for a large class of values of p and q.

3.1 Algorithm

Step 1: Partition the $p(q-1)$ vertices into p sets of size $(q-1)$ each. That is, if the vertices are numbered as $0, 1, 2, \cdots, q-2, q-1, q, \cdots, 2q-3, 2q-2, 2q-1, \cdots, 3q-4, \cdots, pq-p$, partition the vertices as $V_i = \{i(q-1), i(q-1)+1, \cdots, i(q-1)+(q-2)\}$, for $i = 0, 1, 2, \cdots, p-1$.

Step 2: For every $s, t \in \{0, 1, 2, ..., p-1\}$, and for every pair of vertices, say, $i \in V_s$ and $j \in V_t$, color the edge (i, j) with the color $((s + t - 2) \bmod p, (i + j - 2) \bmod q)$.

Observation 3.1 *The number of colors used by the algorithm is pq.*

In the following, we argue its correctness.

Claim 3.2 *The coloring provided by the algorithm is proper.*

Proof. Assume the contrary. That is there exist edges (u, v) and (u, w) with the same color. Suppose $u, v,$ and w belong to vertex sets, say, $V_i, V_j,$ and V_k respectively. By the coloring principle the edge (u, v) receives the color $((i + j - 2) \bmod p, (u + v - 2) \bmod q)$ and the edge (u, w) receives the color $((i + k - 2) \bmod p, (u + w - 2) \bmod q)$.

Case 1: $j = k$ Then, since the color of the edges (u, v) and (u, w) is same, we have

$$((i + j - 2) \bmod p, (u + v - 2) \bmod q) = ((i + j - 2) \bmod p, (u + w - 2) \bmod q)$$

This implies that

$$(u + v - 2) \bmod q = (u + w - 2) \bmod q,$$

which in turn implies

$$(v - w) \equiv 0 \bmod q.$$

$0 \leq v, w \leq q - 1$ and q is a prime. So q cannot divide $v - w$ unless $v = w$ which is a contradiction.

Case 2: $j \neq k$ Then it is required that

$$((i + j - 2) \bmod p, (u + v - 2) \bmod q) = ((i + k - 2) \bmod p, (u + w - 2) \bmod q).$$

This implies

$$(i + j - 2) \bmod p = (i + k - 2) \bmod p \text{ and } (u + v - 2) \bmod q = (u + w - 2) \bmod q.$$

Consider

$$(i + j - 2) \bmod p = (i + k - 2) \bmod p.$$

This implies

$$(j - k) \equiv 0 \bmod p.$$

But this cannot be satisifed as $0 \leq j, k \leq p - 1$ and $j \neq k$ by assumption. So $j = k$. Hence, we arrive at a contradiction in this case also.

Hence the Claim. □

Claim 3.3 *The union of any two color classes will not have an even length cycle.*

Proof. Consider a bichromatic path of edges colored with colors (a, b) and (c, d) and starting from a vertex $u \in V_i$ for some $i \in \{0, 1, 2, ..., p - 1\}$. Consider the edge colored (a, b) and incident at u. Then, by the coloring principle, the other vertex of this edge is $(b - u + 2) \bmod q \in V_{(a-i+2) \bmod p}$. Now consider the edge colored (c, d) and incident at $(b - u + 2) \bmod q$ of $V_{(a-i+2) \bmod p}$. Then, again by the coloring principle, the other vertex of this edge is $(a - b + u) \bmod q \in V_{(c-a+i) \bmod p}$. Following similar argument, the other vertices on this bichromatic path can be calculated to be $(2b - d - u + 2) \bmod q \in V_{(2a-c-i+2) \bmod p}$, $(2d - 2b + u) \bmod q \in V_{(2c-2a+i) \bmod p}$, $(3b - 2d - u + 2) \bmod q \in V_{(3a-2c-i+2) \bmod p}$, $(3d - 3b + u) \bmod q \in V_{(3c-3a+i) \bmod p}$, etc. In general, the vertices on this path can be expressed as $(\ell b - (\ell - 1)d - u + 2) \bmod q \in V_{(\ell a-(\ell-1)c-u+2) \bmod p}$ and $(\ell d - \ell b + u) \bmod q \in V_{(\ell c-\ell a+s) \bmod p}$, where ℓ is the length of the bichromatic path. For this bichromatic path to be a bichromatic cycle of even length, we need

$$u = (\ell d - \ell b + u) \bmod q \quad \text{and} \quad i = (\ell c - \ell a + i) \bmod p.$$

That is

$$u = \ell d - \ell b + u + k_1 q, \quad \text{for some integer } k_1 \;,$$

and

$$i = \ell c - \ell a + i + k_2 p, \quad \text{for some integer } k_2.$$

These in turn imply

$$\ell(d-b) \equiv 0 \bmod q \ \text{ and } \ \ell(c-a) \equiv 0 \bmod p.$$

For the above two conditions to be satisfied at the same time, notice that q cannot divide $d-b$ and similarly p cannot divide $c-a$ as $0 \leq b, d \leq q-1$ and $0 \leq a, c \leq p-1$ and both p, q are prime. So q has to divide ℓ and also p has to divide ℓ. But, ℓ, being the length of the cycle can be at most $p(q-1)$. For both p and q to divide ℓ, ℓ must be at least pq. Hence a contradiction arises implying that no bichromatic cycle can exist. □

4 Algorithm for Complete Graphs with $p(q-1)(r-1)$ Vertices

The previous algorithm for complete graphs of order $p(q-1)$ is hereby extended to complete graphs of order $p(q-1)(r-1)$. The resulting algorithm uses pqr colors. It may be seen that this is an improvement over that of Alon et al. [2] for certain values of p, q, and r.

4.1 Algorithm

Step 1: Partition the $p(q-1)(r-1)$ vertices into $p(q-1)$ sets of size $(r-1)$ each. That is, if the vertices are numbered as $0, 1, 2, \cdots, [p(q-1)(r-1)-1]$, partition the vertices as $V_{sx} = \{[(s(q-1)+x)(r-1)], [(s(q-1)+x)(r-1)+1], \cdots, [(s(q-1)+x)(r-1)+(r-2)]\},\ s = 0, 1, 2, \cdots, p-1$ and $x = 0, 1, 2, \cdots, q-2$.

Step 2: For every pair of vertices, say $i \in V_{sx}$ and $j \in V_{ty}$, color the edge (i, j) with the color $((s+t-2) \bmod p, (x+y-2) \bmod q, (i+j-2) \bmod r)$.

Observation 4.1 *The algorithm may be seen to use pqr colors.*

Claim 4.2 *The coloring provided by the algorithm is proper.*

Proof. Assume the contrary. That is, there exist edges incident at vertex u, say (u, v) and (u, w), with the same color. Suppose u, v, and w belong to V_{fg}, V_{hi}, and V_{jk} respectively. By the coloring principle, edge (u, v) receives $((f+h-2) \bmod p, (g+i-2) \bmod q, (u+v-2) \bmod r)$ and the edge (u, w) receives $((f+j-2) \bmod p, (g+k-2) \bmod q, (u+w-2) \bmod r)$. From the assumption, this implies that $(f+h-2) \bmod p = (f+j-2) \bmod p$, $(g+i-2) \bmod q = (g+k-2) \bmod q$, and $(u+v-2) \bmod r = (u+w-2) \bmod r$. These in turn imply:

$$h - j + k_1 p = 0 \quad \text{for some integer } k_1,$$

$$i - k + k_2 q = 0 \quad \text{for some integer } k_2,$$

and

$$v - w + k_3 r = 0 \quad \text{for some integer } k_3.$$

But $0 \leq h, j \leq p-1$, $0 \leq i, k \leq q-2$, and $0 \leq j, k \leq r-2$, and p, q, and r are prime numbers. So, it follows that $h = j, i = k$, and $v = w$. That is the edges that receive the same color must be one and the same. □

Claim 4.3 *The union of any two color classes will not have an even length cycle.*

Proof. Consider a bichromatic path of edges colored with colors $(a, b, c), (d, e, f)$ and starting from a vertex $u \in V_{ij}$ for some $i \in 0, 1, 2, \cdots, p-1$ and $j \in 0, 1, 2, \cdots, q-2$. Also, consider the edge colored (a, b, c) and incident at u. Then, by the coloring principle, the other vertex of this edge is $(c - u + 2) \bmod r \in V_{(a-i+2) \bmod p, (b-j+2) \bmod q}$. Now consider the edge colored (d, e, f) and incident at $(c - u + 2) \bmod r$ of $V_{(a-i+2) \bmod p, (b-j+2) \bmod q}$. Then, again, by the coloring principle, the other vertex of this edge is $(f - c + u) \bmod r \in V_{(d-a+i) \bmod p, (e-b+j) \bmod q}$. Following similar argument, the other vertices on this bichromatic path can be calculated to be:

$$\begin{aligned}
&(2c - f - u + 2) \bmod r \in V_{(2a-d-i+2) \bmod p, (2b-e-j+2) \bmod q},\\
&(2f - 2c + u) \bmod r \in V_{(2d-2a+i) \bmod p, (2e-2b+j) \bmod q},\\
&(3c - 2f - u + 2) \bmod r \in V_{(3a-2d-i+2) \bmod p, (3b-2e-j+2) \bmod q}, \text{ and}\\
&(3f - 3c + u) \bmod r \in V_{(3d-3a+i) \bmod p, (3e-3b+j) \bmod q}
\end{aligned}$$

and so on. In general, the vertices on this path can be expressed as:

$$\begin{aligned}
&(\ell c - (\ell-1)f - u + 2) \bmod r \in V_{(\ell a-(\ell-1)d-i+2) \bmod p, (\ell b-(\ell-1)f-j+2) \bmod q} \text{ and}\\
&(\ell f - \ell c + u) \bmod r \in V_{(\ell d-\ell a+i) \bmod p, (\ell e-\ell b+j) \bmod q}
\end{aligned}$$

where ℓ is the length of the bichromatic path. For this bichromatic path to be a bichromatic cycle of even length, we need $(\ell f - \ell c + u) \bmod r = u, (\ell d - \ell a + i) \bmod p = i, (\ell e - \ell b + j) \bmod q = j$. But these imply

$$\begin{aligned}
\ell f - \ell c + u + k_1 r = u \quad &\text{for some integer } k_1,\\
\ell d - \ell a + i + k_2 p = i \quad &\text{for some integer } k_2,\\
\ell e - \ell b + j + k_3 q = j \quad &\text{for some integer } k_3.
\end{aligned}$$

These, in turn imply

$$\begin{aligned}
\ell f - \ell c &\equiv 0 \bmod r,\\
\ell d - \ell a &\equiv 0 \bmod p,\\
\ell e - \ell b &\equiv 0 \bmod q.
\end{aligned}$$

For all the above three conditions to be satisfied simultaneously, notice that r cannot divide $f - c$, p cannot divide $d - a$, and q cannot divide $e - b$ as p, q, r are prime and $0 \le c, f \le r - 1$, $0 \le d, a \le p - 1$, and $0 \le b, e \le q - 1$. So p, q, and r have to divide ℓ simultaneously. However, this is not possible as $\ell \le p(q-1)(r-1)$ and the smallest integer divisible by all of p, q, and r is pqr. Hence, we arrive at a contradiction meaning that no bichromatic cycles can exist. $\square$

Remark 4.4 *One may be tempted to extend the approach for complete graphs of order $p(q-1)(r-1)(s-1)$, s also prime, and beyond. However, as the smallest prime greater than an integer n is known to be within $n + O(n^{2/3})$ [14], the approach will not result in any saving in the number of colors used, at least asymptotically.*

5 Algorithm for $K_{n,n}$, where $n = (p-1) \cdot (q-1)$.

In this section, we describe our algorithm to color a complete bipartite graph with $n = (p-1)(q-1)$ vertices on each side of the partition. Our algorithm uses the algorithm of Alon et al. [2] to color $K_{p-1,p-1}$ using p colors. Notice that their approach can be used to color any $K_{n,n}$ but the number of colors increase to the smallest prime greater than n. It is known that for an integer n, the smallest prime greater than n is in $n + O(n^{2/3})$. Our algorithm views $K_{n,n}$ as a K_{q-1} with each edge being $p-1$ multi-edges and uses pq colors. This results in a saving of number of colors used. Our algorithm is as follows.

First, we denote the complete bipartite graph $K_{n,n}$ as $G = (V \cup W, E)$ with $|V| = |W| = n$ and $E = \{(u, v) \mid u \in V, v \in W\}$. The algorithm is given below.

Algorithm Color($K_{n,n}$)
1. Partition the vertices in each side of the partition into $p-1$ sets of $q-1$ vertices each. Let $V_i \subset V$ be the set of vertices numbered $(i-1)(q-1)+1$ to $i(q-1)$ for $i = 1, 2, \cdots, p-1$. Similarly, let $W^i \subset W$ be the set of vertices numbered $(i-1)(q-1)+1$ to $i(q-1)$ for $i = 1, 2, \cdots, p-1$.

2. Color the edge (u, v) with $u \in V_1^i$ and $v \in V_2^j$ with the color $(u + v - 2 \bmod q, i + j - 2 \bmod p)$.
End Algorithm.

It can be easily seen that the number of colors used by the above algorithm is pq. The idea behind the above algorithm is that when all the vertices in each V_i, W_i are treated as a single vertex, the resulting graph resembles a $K_{p-1,p-1}$ which can be colored using p colors [2]. Now, this color value can be interpreted as a color class of q colors as for each edge in the $K_{p-1,p-1}$ corresponds to $q-1$ multiedges. In the following, we show that the coloring is proper and does not have any bichromatic cycles.

Claim 5.1 *The coloring obtained is proper.*

Proof. On the contrary, assume that two edges have the same color. Let the edges be (u, v) and (u, w) with $u \in V_i, v \in W_j$, and $w \in W_k$ for $0 \le i, j, k \le p-1$ and $0 \le u, v, w \le q-1$. The color of (u, v) is $(u+v-2 \bmod q, i+j-2 \bmod p)$ and the color assigned to (u, w) is $(u+w-2 \bmod q, i+k-2 \bmod p)$. Now we make a case distinction as follows.

Case $j = k$ In this case, the above conditions imply that we need to have $u + v - 2 \bmod q \equiv u + w - 2 \bmod q$ which implies that $v - w \equiv 0 \bmod q$. But, both v and w can $0 \le v, w \le q-1$ and q is a prime. Hence, we arrive at a contradiction.

Case $j \ne k$ In this case, we require that $i+j-2 \equiv i+k-2 \bmod p$ and $u+v-2 \equiv u+w-2 \bmod q$. These in turn imply that $j - k \equiv 0 \bmod p$ and $v - w \equiv 0 \bmod q$. Now, notice that $0 \le j, k \le p-1$ and that p is a prime. So, unless $j = k$, $j - k \equiv 0 \bmod q$ cannot be satisfied. Similarly, $0 \le v, w \le q-1$ and q is a prime. So $v - w \equiv 0 \bmod q$ cannot satisfied unless $v = w$. In this case too, we arrive at a contradiction.

Hence the coloring obtained is proper. □

Claim 5.2 *The coloring does not induce any bichromatic cycles.*

Proof. Consider a bichromatic path starting at vertex $u \in V_i$ with edges alternating between colors (a, b) and (c, d) where $0 \le u \le q-1$ and $0 \le i \le p-1$. The other end point of this edge colored (a, b) can be seen to be of the form $(a - u + 2) \bmod q \in W_{(b-i+2) \bmod p}$. By our assumption, there is an edge colored (c, d) from this vertex. The other endpoint of this edge can be calculated to be $(c - a + u) \bmod q \in V_{(d-b+i) \bmod p}$. Extending this line of argument, we have that, the last edge has endpoints of the form $((\ell-1)(c-a)+u) \bmod q \in V_{(\ell-1)(d-b)+i) \bmod p}$, where ℓ is the length of the path. For this path to be a bichromatic cycle, we need that:

$$\begin{aligned}(\ell-1)(c-a)+u \bmod q &= u \\ (\ell-1)(d-b)+i \bmod p &= i\end{aligned}$$

For the above set of equations to hold, we need that $(\ell-1)(c-a) = k_1 \cdot q$ and $(\ell-1)(d-b) = k_2 \cdot p$ for some integers k_1, k_2. But, note that $0 \leq \ell \leq (p-1)(q-1)$ and $0 \leq a, c \leq q-1$, and $0 \leq b, d \leq p-1$ by construction. So the above equations cannot hold simultaneously as p and q are prime. Hence the proof. □

Remark 5.3 *One can think of extending this approach to color $K_{n,n}$ with $n = (p-1)(q-1)(r-1)$. However, no benefit can be gained, at least asymptotically, as the smallest prime greater than n shall be smaller than the number of colors used by the above apporach [14].*

6 Conclusions

In this paper, we have presented several simple algorithms for acyclically coloring complete graphs and complete bipartite graphs. Our algorithms improve the state of the art in some cases and are explicit. For example, when using our algorithm for coloring K_{510}, with $p = 5$ and $q = 103$, both prime, our algorithm from Section 3 requires 515 colors. The optimal number of colors needed to acyclically color K_{510} is 511 whereas using the algorithm of Alon et al. [2] requires 521 colors, which is the smallest prime after 510. Similarly, one can find instances where the number of colors used by our algorithms are close to the optimal value. It remains to be seen whether the proposed algorithms apply to any wider class of graphs.

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