Black Scholes Equation

Evolution of Portfolio value

- Consider you have a portfolio which at any time t has a value denoted by X(t).
- This portfolio invests
  a. In a money market(bank) account that pays a constant rate of interest r
  b. In a stock modeled by the Geometric Brownian Motion (GBM)

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t).$$

The portfolio $X(t)$

- At each t, suppose the investor holds $\Delta(t)$ shares of the above stock.
- $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$.
- The remainder of the portfolio value, $X(t)- \Delta(t)S(t)$ is invested in the money market.

Change in the portfolio value

- The change is the investor's portfolio at each time t is due to
  1. Capital gains Δ(t)dS(t) on the stock position
  2. Interest on the bank account r(X(t)- Δ(t)S(t))dt
- Therefore

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\,dt \\ &= \Delta(t)\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big) + r\big(X(t) - \Delta(t)S(t)\big)\,dt \\ &= rX(t)\,dt + \Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t). \end{aligned}$$

Discounted portfolio value

- If we are interested in the present value of the future change in the portfolio

$$
\begin{aligned}
&d\big(e^{-rt}X(t)\big)\\
&= df\big(t, X(t)\big)\\
&= f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t)\\
&= -re^{-rt}X(t)\,dt + e^{-rt}\,dX(t)\\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)\,dt + \Delta(t)\sigma e^{-rt}S(t)\,dW(t)
\end{aligned}
$$

European Option

- A European option gives the option holder the right but not obligation to buy (call) or sell (put) an asset at a prespecified price (called the strike).
- A call option pays $(S(T)-K)^+$ at time T.
- The value of this option at any time (t) would depend upon
  - The time to expiry, i.e. (T-t)
  - Value of the stock, i.e. S(t)
  - Agreed strike price (K)
  - It also depends upon the model parameters (r, sigma) which are assumed to be constant
- Therefore we can use c(t,x) to denote the value of a call option, where x at time t is S(t).

Change in option price

- Our goal is to determine a function c(t,x), so that if we plug in the stock price at any time we get the corresponding option price.

$$
\begin{aligned}
& dc\big(t, S(t)\big) \\
& = c_t\big(t, S(t)\big)\, dt + c_x\big(t, S(t)\big)\, dS(t) + \frac{1}{2} c_{xx}\big(t, S(t)\big)\, dS(t)\, dS(t) \\
& = c_t\big(t, S(t)\big)\, dt + c_x\big(t, S(t)\big)\big(\alpha S(t)\, dt + \sigma S(t)\, dW(t)\big) \\
& \qquad + \frac{1}{2} c_{xx}\big(t, S(t)\big) \sigma^2 S^2(t)\, dt \\
& = \Big[c_t\big(t, S(t)\big) + \alpha S(t) c_x\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t) c_{xx}\big(t, S(t)\big)\Big]\, dt \\
& \qquad + \sigma S(t) c_x\big(t, S(t)\big)\, dW(t).
\end{aligned}
$$

Change in the discounted option price

- Using the Ito's expansion for $f(t,x) = e^{-rt}x$, where x is c(t,S(t)).

$$
\begin{aligned}
&d\big(e^{-rt}c(t,S(t))\big)\\
&\quad= df\big(t,c(t,S(t))\big)\\
&\quad= f_t\big(t,c(t,S(t))\big)\,dt + f_x\big(t,c(t,S(t))\big)\,dc(t,S(t))\\
&\qquad\quad+\frac{1}{2}f_{xx}\big(t,c(t,S(t))\big)\,dc(t,S(t))\,dc(t,S(t))\\
&\quad= -re^{-rt}c(t,S(t))\,dt + e^{-rt}\,dc(t,S(t))\\
&\quad= e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t))\\
&\qquad\quad+\frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]\,dt + e^{-rt}\sigma S(t)c_x(t,S(t))dW(t).
\end{aligned}
$$

Hedging an option

- When you sell an option you start with an initial capital X(0) and invest it in stock and money market account
- The aim of the portfolio is to ideally at each time perfectly replicate the option value, c(t,S(t)).
- This will happen if the present value of the future portolio is always equal to the present value of the option, i.e. for all t
  - $e^{-rt}X(t) = e^{-rt}c\big(t, S(t)\big)$
- The above equality implies

$$d\big(e^{-rt}X(t)\big) = d\big(e^{-rt}c(t, S(t))\big) \text{ for all } t \in [0, T)$$

Hedging continued

- Substituting the LHS and RHS, we have

$$\begin{aligned}&\Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t)\\ &= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt\\ &\quad+\sigma S(t)c_x(t,S(t))\,dW(t). \qquad (4.5.10)\end{aligned}$$

- First looking at dW(t) term:

$$\Delta(t) = c_x(t,S(t)) \text{ for all } t \in [0,T).$$

- This is the *delta-hedging* rule. At each time prior to the expiration, the number of shares held in the hedge portfolio is equal to the partial derivative with respect to the stock price of the option value at that time.
- c_x(t,S(t)) is called the delta of the option.

Equating the dt term

- We next equate the dt terms to obtain

$$(\alpha - r)S(t)c_x(t, S(t))$$
$$= -rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))$$
$$\text{for all } t \in [0, T). \quad (4.5.12)$$

- The alpha terms cancel out leaving

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, x) = rc(t, x) \text{ for all } t \in [0, T), \ x \geq 0,$$

- c(t,x) is the solution of the above pde

Solution to the pde

- The pde is a backward parabolic type.
- For such an equation, in addition to the terminal condition one needs boundary conditions.
  - The terminal condition is ?
  - The boundary condition at x= 0 ?
  - The boundary condition at x goes to infinity ?
- The solution of the BS equation with these boundary and terminal conditions is

Black Scholes equation

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big),\ 0 \le t < T,\ x > 0,$$

- Where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right],$$

- And N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

- Sometimes the notation used is

$$\mathrm{BSM}(\tau,x;K,r,\sigma) = xN\big(d_+(\tau,x)\big) - Ke^{-r\tau}N\big(d_-(\tau,x)\big),$$

Forward contract

- A forward contract on a stock with delivery K obligates its holder to buy one stock at expiration T for K.
- At T the value of the contract is S(T)-K.
- Let f(t,x) denote the value of this contract at any time t in [0, T].
- We can show that
  - $f(t,x) = x - e^{(-r(T-t))}K$.
- This can be shown using the no-arbitrage argument.

Forward contract

- If you sell forward contract at time zero for
  - $f(0,S(0)) = S(0)-e^{-r(T)}K$
- You can use the proceed to buy one unit of stock (price S(0))
- You will have to borrow $e^{-rT}K$ from money market.
- At the expiration he has one unit of stock, i.e. S(T), owes the bank an amount K
- He needs to give the buyer of the forward contract a stock for an amount K.
- As he is able to replicate the payoff of the forward contract with this portfolio whose value at time t is
  - $S(t)-e^{-r(T-t)}K$, this should be the forward price at any time t.

Put Call Parity

- At any time

$$x - K = (x - K)^+ - (K - x)^+$$

- The payoff of a forward contract at T is equal to

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T));$$

- It should also then agree at any time t

$$f(t, x) = c(t, x) - p(t, x), \quad x \geq 0,\ 0 \leq t \leq T.$$