Stochastic Calculus

Quadratic Variation

- Before we look at Quadratic Variation let us consider first order variation (FOV).
- FOV gives the amount of up down movement of a function between 0 and T, with down moves adding to rather than subtracting.

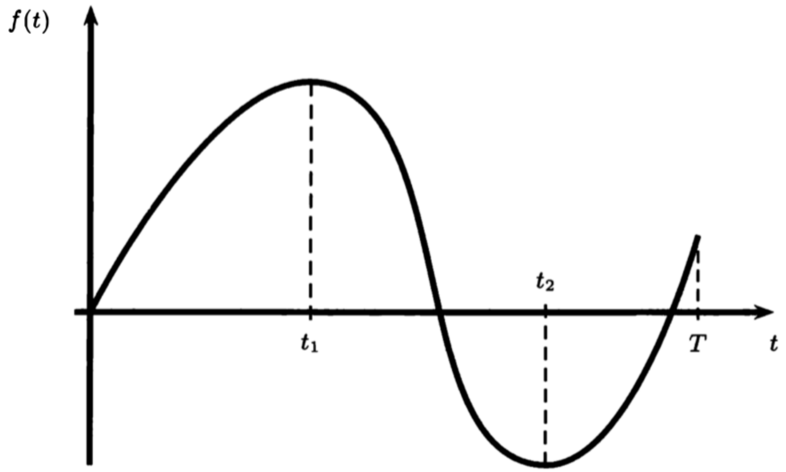

Fig. 3.4.1. Computing the first-order variation.

$$\begin{aligned}\mathrm{FV}_T(f) &= [f(t_1) - f(0)] - [f(t_2) - f(t_1)] + [f(T) - f(t_2)] \\ &= \int_0^{t_1} f'(t)\,dt + \int_{t_1}^{t_2} (-f'(t))\,dt + \int_{t_2}^{T} f'(t)\,dt \\ &= \int_0^T |f'(t)|\,dt.\end{aligned}$$

FOV

- In general you first partition [0, T] as

$$0 = t_0 < t_1 < \cdots < t_n = T. \qquad \Pi = \{t_0, t_1, \ldots, t_n\}$$

- The maximum step size of the partition is denoted as

$$\|\Pi\| = \max_{j=0,\ldots,n-1}(t_{j+1} - t_j)$$

- Then

$$\mathrm{FV}_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

FOV continued

- We use Mean Value Theorem

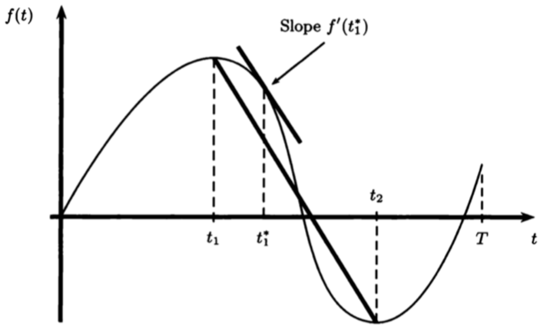

Fig. 3.4.2. Mean Value Theorem.

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = f'(t_j^*).$$

- Therefore, we can write

$$\mathrm{FV}_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j) = \int_0^T |f'(t)|\, dt,$$

Quadratic Variation

- According to definition of Quadratic variation

Definition 3.4.1. *Let $f(t)$ be a function defined for $0 \leq t \leq T$. The* quadratic variation *of f up to time T is*

$$[f,f](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2, \qquad (3.4.5)$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$.

Quadratic variation of f (continuous and differentiable)

- Using MVT we have

$$\sum_{j=0}^{n-1}[f(t_{j+1})-f(t_j)]^2=\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)^2\le\|\Pi\|\cdot\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j),$$

- Which translates to

$$\begin{aligned}
[f,f](T) &\le \lim_{\|\Pi\|\to 0}\left[\|\Pi\|\cdot\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)\right]\\
&= \lim_{\|\Pi\|\to 0}\|\Pi\|\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)\\
&= \lim_{\|\Pi\|\to 0}\|\Pi\|\cdot\int_0^T|f'(t)|^2\,dt \;=\; 0.
\end{aligned}$$

Quadratic variation of W (Brownian Motion)

- Define sampled Quadratic variation for a partition of [0,T] as

$$Q_\Pi = \sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)^2.$$

- Prove that the above **random variable** converges to T as $\|\Pi\| \to 0.$
- This can be shown if the expected value of Q is T and its variance converges to 0.

Definition of Brownian Motion

- Definition of Brownian Motion:

Definition 3.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$, is a* Brownian motion *if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \ldots,\ W(t_m) - W(t_{m-1}) \quad (3.3.1)$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}\big[W(t_{i+1}) - W(t_i)\big] = 0, \quad (3.3.2)$$

$$\mathrm{Var}\big[W(t_{i+1}) - W(t_i)\big] = t_{i+1} - t_i. \quad (3.3.3)$$

In order to prove QV of W

- We first use:

$$\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \mathrm{Var}\left[W(t_{j+1}) - W(t_j)\right] = t_{j+1} - t_j$$

- Therefore,

$$\mathbb{E}Q_{\Pi} = \sum_{j=0}^{n-1} \mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \sum_{j=0}^{n-1}(t_{j+1} - t_j) = T,$$

- Next we determine that variance of Q

Variance of Q

$$
\begin{aligned}
&\operatorname{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] \\
&= \mathbb{E}\left[\left(\left(W(t_{j+1}) - W(t_j)\right)^2 - (t_{j+1} - t_j)\right)^2\right] \\
&= \mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right] - 2(t_{j+1} - t_j)\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] \\
&\qquad + (t_{j+1} - t_j)^2.
\end{aligned}
$$

- The 4^{th} moment of normal random variable with zero mean is three times its variance squared (home work exercise)

$$
\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right] = 3(t_{j+1} - t_j)^2,
$$

- Therefore

$$
\begin{aligned}
\operatorname{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] &= 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 \\
&= 2(t_{j+1} - t_j)^2, \qquad (3.4.7)
\end{aligned}
$$

Continued ..

$$\operatorname{Var}(Q_\Pi) = \sum_{j=0}^{n-1} \operatorname{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2$$

$$\leq \sum_{j=0}^{n-1} 2\|\Pi\|(t_{j+1} - t_j) = 2\|\Pi\|T.$$

- Therefore

$$\lim_{\|\Pi\|\to 0} \operatorname{Var}(Q_\Pi) = 0,$$

$$\lim_{\|\Pi\|\to 0} Q_\Pi = \mathbb{E}Q_\Pi = T.$$

Implication

- We have

$$\mathbb{E}\big[\big(W(t_{j+1}) - W(t_j)\big)^2\big] = t_{j+1} - t_j$$

$$\mathrm{Var}\big[\big(W(t_{j+1}) - W(t_j)\big)^2\big] = 2(t_{j+1} - t_j)^2.$$

- When t_{j+1}-t_j is small then $(t_{j+1}$-$t_j)^2$ is very small

$$\big(W(t_{j+1}) - W(t_j)\big)^2 \approx t_{j+1} - t_j.$$

- Or E[ΔWΔW] ≈ Δt

$$dW(t)\,dW(t) = dt.$$

Ito’s Integral

- For a normal integral

$$\int_0^T \Delta(t)\, dg(t) = \int_0^T \Delta(t) g'(t)\, dt,$$

- However, W(t) is not differentiable, so

$$\int_0^T \Delta(t)\, dW(t).$$

- You need to use Ito’s integral

Constructing the Ito's integral

- Figure shows a simple path of Δ(t)
- Then define

$$I(t) = \Delta(t_0)[W(t) - W(t_0)] \; = \; \Delta(0)W(t), \quad 0 \le t \le t_1,$$
$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \le t \le t_2,$$
$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \le t \le t_3,$$

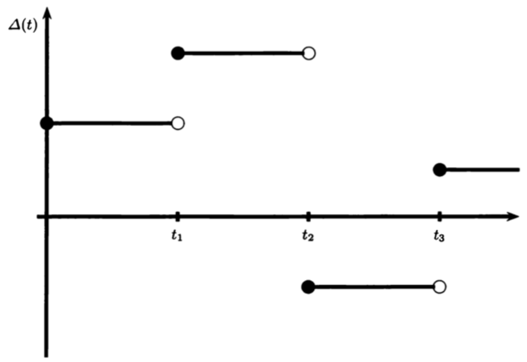

Fig. 4.2.1. A path of a simple process.

- Ito's integral of the process is

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)].$$

$$I(t) = \int_0^t \Delta(u)\, dW(u).$$

Properties of Ito's integral

Theorem 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \le t \le T$, be an adapted stochastic process that satisfies (4.3.1). Then $I(t) = \int_0^t \Delta(u)\, dW(u)$ defined by (4.3.3) has the following properties.*

(i) **(Continuity)** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*

(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)\, dW(u)$ and $J(t) = \int_0^t \Gamma(u)\, dW(u)$, then $I(t) \pm J(t) = \int_0^t \big(\Delta(u) \pm \Gamma(u)\big)\, dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)\, dW(u)$.*

(iv) **(Martingale)** *$I(t)$ is a martingale.*

(v) **(Itô isometry)** $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\, du$.

(vi) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u)\, du$.

Ito Doeblin Formula

- If W was differentiable, we could write

$$df(W(t)) = f'(W(t))\,W'(t)\,dt = f'(W(t))\,dW(t).$$

- However, as W has non zero quadratic variation, the correct formula is

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2}f''(W(t))\,dt.$$

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$f\big(T,W(T)\big) = f\big(0,W(0)\big) + \int_0^T f_t\big(t,W(t)\big)\,dt + \int_0^T f_x\big(t,W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt. \quad (4.4.3)$$

Proof

- Let x_j, x_{j+1} be numbers, then by Taylor expansion:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2.$$

- We want to find f(W(T))-f(W(0))
- Divide [0,T] into intervals. We look at one such interval $[t_j, t_{j+1}]$

$$\begin{aligned} f(W(T)) - f(W(0)) &= \sum_{j=0}^{n-1} \left[f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) \right] \\ &= \sum_{j=0}^{n-1} f'(W(t_j)) \big[W(t_{j+1}) - W(t_j)\big] \\ &\quad + \frac{1}{2} \sum_{j=0}^{n-1} f''(W(t_j)) \big[W(t_{j+1}) - W(t_j)\big]^2. \end{aligned}$$