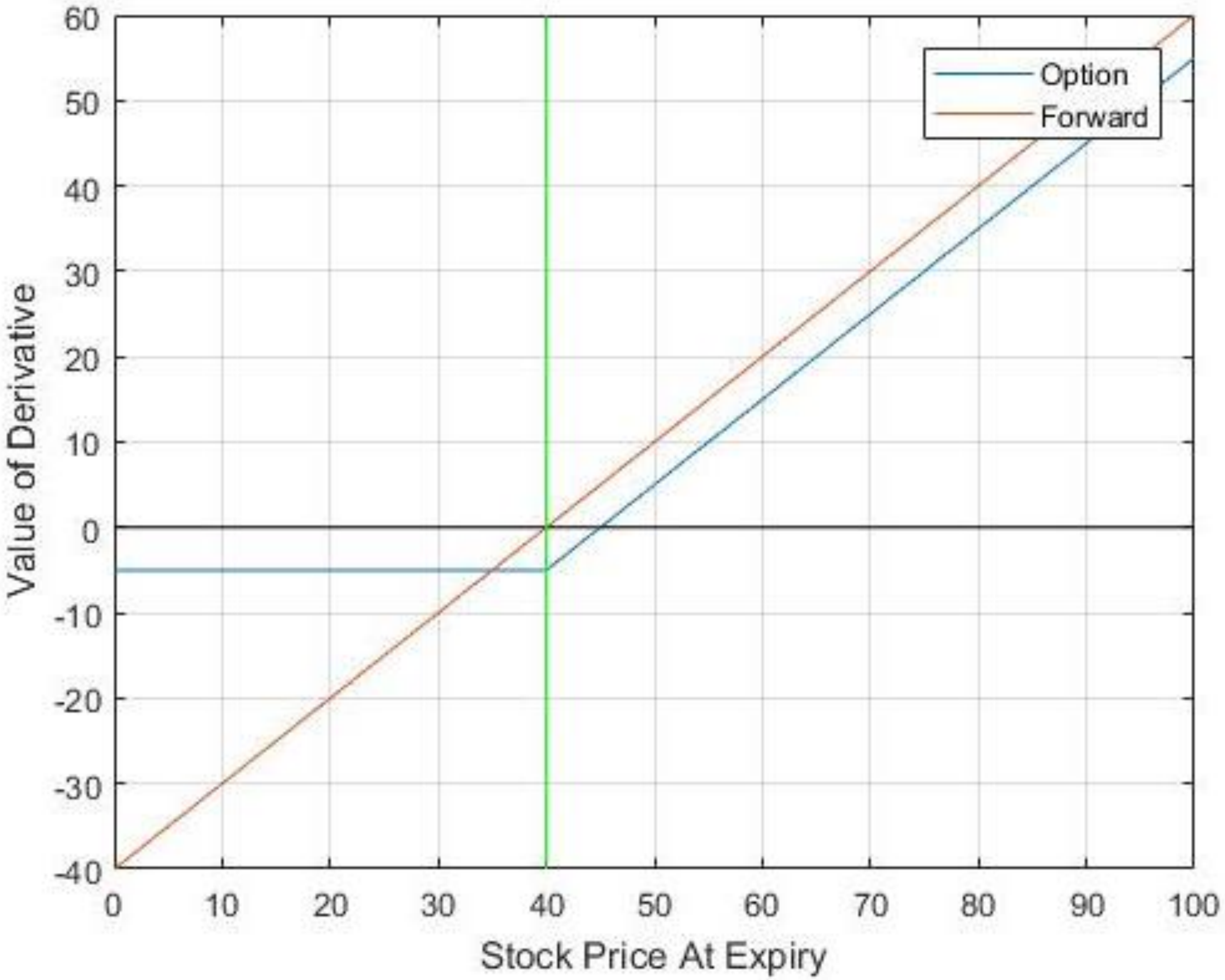
Introduction to Stochastic Calculus

ACM Winter School on AI and Finance

IIIT Hyderabad

Futures and Options

- A futures/forward contract gives the holder the obligation to buy or sell at a certain price
- An option gives the holder the right to buy or sell at a certain price



INDIGO **5,367.50** **-69.00(-1.27 %)**

30-Dec-2025 Or Select Strike Price | Refresh | Clear

Underlying : **INDIGO 5,367.50**
As on : 05-Dec-2025 15:30:00 IST

Calls											Puts									
OI	Chng In OI	Volume	IV	LTP	Chng	BID QTY	BID	ASK	ASK QTY	Strike	BID QTY	BID	ASK	ASK QTY	Chng	LTP	IV	Volume	Chng In OI	OI
63	63	308	25.66	301.40	-323.60	150	299.00	305.35	150	5,150.00	150	54.55	55.95	150	19.00	55.35	27.20	5,935	179	309
475	432	2,239	25.24	263.20	-42.95	150	260.80	264.00	300	5,200.00	150	65.05	66.00	2,400	21.55	65.80	26.43	27,156	480	1,84
136	136	1,131	24.59	226.00	-471.85	150	223.75	227.90	150	5,250.00	300	77.65	79.00	300	23.55	78.05	25.66	12,622	415	618
922	896	12,023	23.50	188.75	-39.10	150	188.65	192.75	150	5,300.00	600	94.00	94.65	150	26.30	94.00	25.19	37,356	1,357	2,31
802	777	7,913	23.47	159.60	-36.95	300	158.50	161.85	150	5,350.00	150	112.45	113.95	150	31.85	113.95	24.99	16,897	633	886
3,058	2,682	32,397	23.35	133.00	-29.90	150	131.50	133.40	300	5,400.00	150	133.65	135.85	150	35.25	135.40	24.56	35,524	1,017	2,54
1,326	1,064	17,195	22.97	107.85	-27.25	300	107.30	108.95	300	5,450.00	300	159.00	162.35	150	40.60	161.40	24.45	9,687	33	686
4,248	2,528	38,173	23.09	88.50	-23.20	900	86.90	88.50	900	5,500.00	300	189.00	191.55	150	46.55	192.50	24.75	10,821	-152	2,17
1,101	499	10,282	22.84	70.00	-18.40	150	69.15	70.90	150	5,550.00	300	218.65	223.65	150	48.45	223.75	24.60	1,956	49	684
3,053	803	21,158	23.07	56.65	-14.10	150	56.25	56.85	1,050	5,600.00	150	253.20	259.50	150	52.80	260.10	24.96	1,518	-97	1,07
661	269	5,512	23.41	46.00	-10.05	150	45.05	46.10	150	5,650.00	150	293.25	298.90	150	55.25	292.40	23.97	93	-12	143

INDIGO 4,986.50 -384.00(-7.15 %)

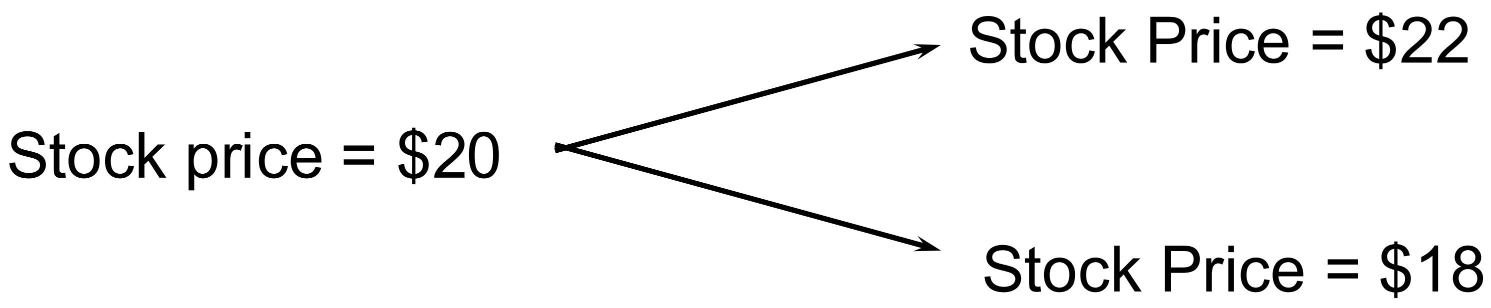
Select Expiry Date | Or | Select Strike Price | Refresh | Clear

Underlying : **INDIGO 4,990.00**
As on : 08-Dec-2025 11:18:21 IST

Calls											Puts										
OI	Chng In OI	Volume	IV	LTP	Chng	BID QTY	BID	ASK	ASK QTY	Strike	BID QTY	BID	ASK	ASK QTY	Chng	LTP	IV	Volume	Chng In OI	OI
25	25	64	48.04	348.00	-588.90	150	346.75	352.20	150	4,800.00	150	130.45	131.05	300	113.40	131.15	43.52	19,306	1,545	2,596
-	-	1	-	322.05	-712.65	150	310.05	319.85	300	4,850.00	150	144.60	145.60	300	125.30	145.50	42.62	5,739	360	523
31	31	51	40.33	280.00	-577.45	150	278.20	280.00	150	4,900.00	150	160.20	160.90	300	137.45	161.00	41.60	19,652	1,725	2,230
36	86	109	39.94	247.75	-697.95	150	246.05	248.15	150	4,950.00	150	178.10	179.10	300	151.70	179.40	40.58	8,413	937	1,108
115	1,067	5,721	38.60	216.05	-213.10	300	215.60	216.95	150	5,000.00	300	197.05	197.80	450	165.15	197.80	39.42	55,424	5,418	8,125
33	598	2,528	37.47	188.35	-199.65	150	187.15	188.25	150	5,050.00	150	218.00	218.80	150	180.25	218.40	38.56	14,018	1,465	1,684
948	1,888	11,372	36.80	160.50	-188.00	150	160.70	161.70	150	5,100.00	450	240.30	241.45	150	196.40	241.85	37.69	31,027	1,245	2,364
270	1,207	8,068	35.91	137.10	-164.30	300	137.10	137.95	150	5,150.00	150	267.00	267.90	150	213.25	267.20	36.65	11,451	369	678
334	3,359	21,919	34.98	115.25	-148.90	150	115.55	116.20	150	5,200.00	300	294.35	296.05	450	231.50	296.00	35.93	18,529	251	2,098
77	641	6,215	34.35	97.10	-130.30	150	97.35	97.90	450	5,250.00	150	324.85	327.05	150	243.15	320.90	35.38	2,393	-266	352

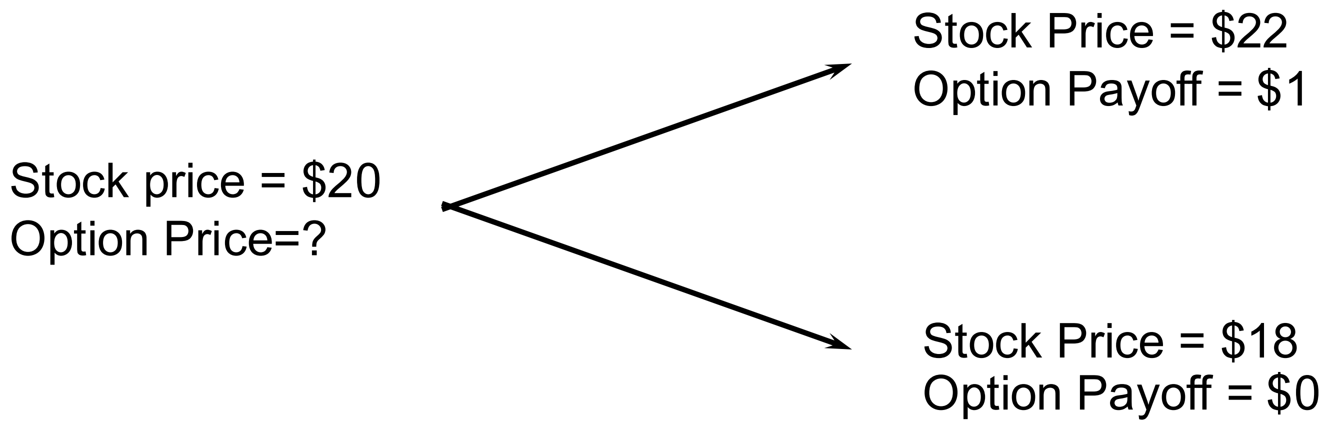
Binomial trees

- A stock price is currently $20
- In 3 months it will be either $22 or $18



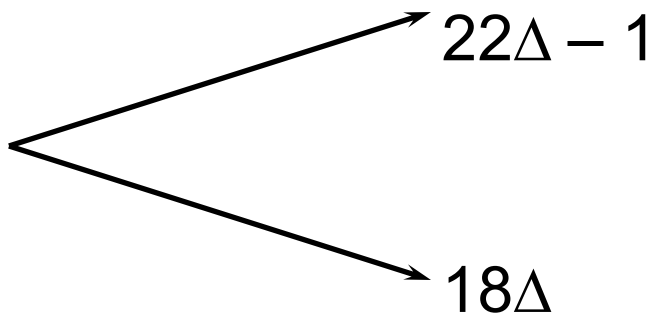
A call option

A 3-month call option on the stock has a strike price of 21.



Making it riskless

- For a portfolio that is long Δ shares and a short 1 call option values are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Value of the portfolio

- The riskless portfolio is:

  long 0.25 shares

  short 1 call option

- The value of the portfolio in 3 months is

  $22 \times 0.25 - 1 = 4.50$

- The value of the portfolio today is

  $4.5e^{-0.04\times0.25} = 4.455$

Valuing the option

- The portfolio that is

  long 0.25 shares

  short 1 option

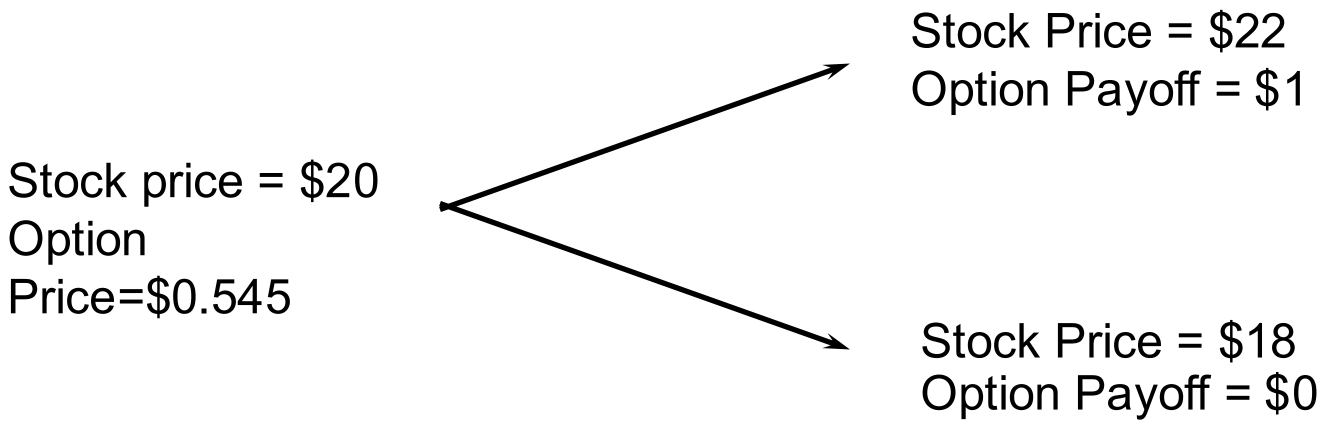
  is worth 4.455 (today)
- The value of the shares held today is

  5.000 (= 0.25 × 20)
- The value of the option is therefore

  5.000 – 4.455 = 0.545

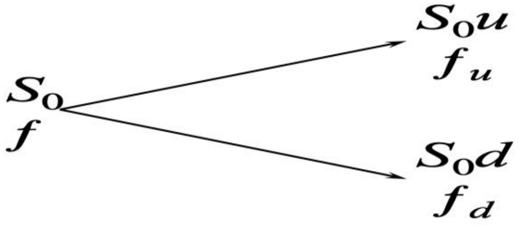
A call option

A 3-month call option on the stock has a strike price of 21.

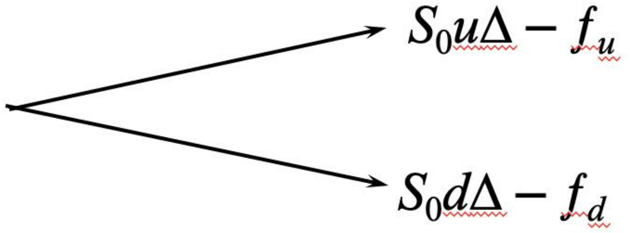


General setting

- A derivative lasts for time T and is dependent on a stock



- Value of a portfolio that is long Δ shares and short 1 derivative:



- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\Delta - f_d$ or $\Delta = \dfrac{f_u - f_d}{S_0u - S_0d}$

Steps

- Value of the portfolio at time T is $S_0 u\Delta - f_u$
- Value of the portfolio today is $(S_0 u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

Substituting for Δ we obtain

$$f = [\,pf_u + (1-p)f_d\,]e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

Tree Parameters for asset paying a dividend yield of q

Parameters p, u, and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world

Mean: $e^{(r-q)\Delta t} = pu + (1-p)d$

Variance: $\sigma^2\Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$

A further condition often imposed is $u = 1/d$

For small Δt

When Δt is small a solution to the equations is

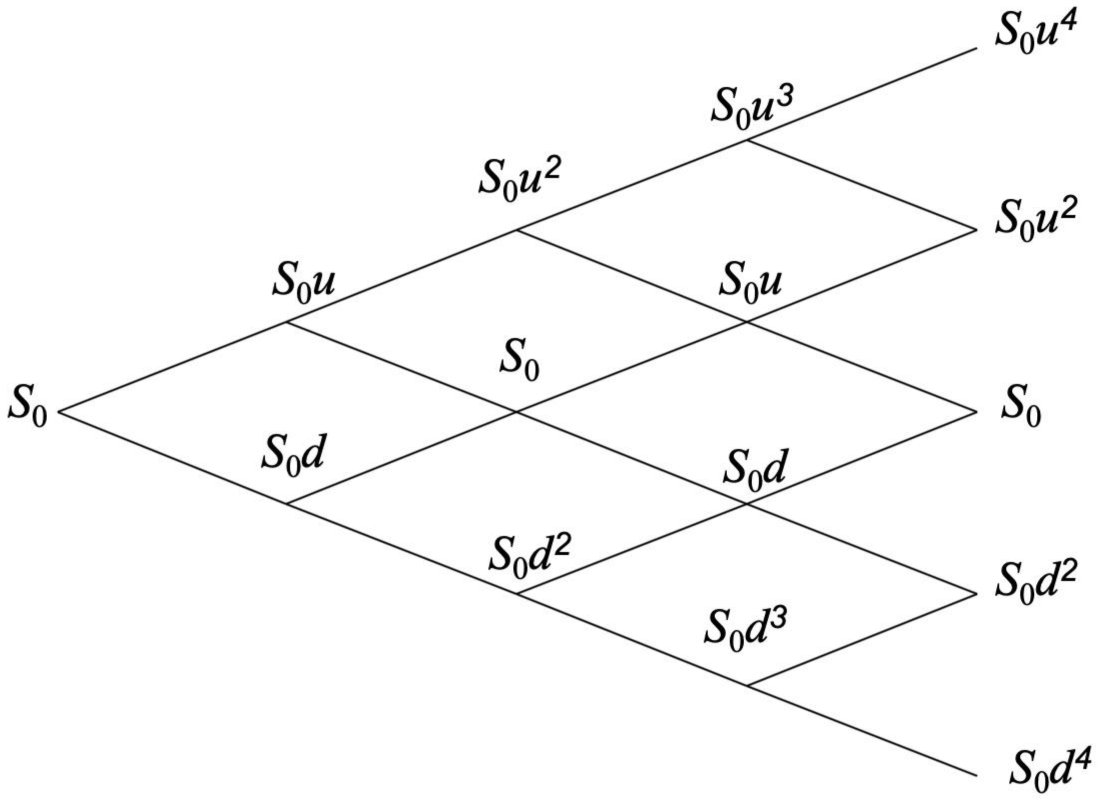
$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{(r-q)\,\Delta t}$$

Tree with more steps



A put Option

S_0 = 50; K = 50; r =10%; σ = 40%;

T = 5 months = 0.4167; Δt = 1 month = 0.0833

In this case

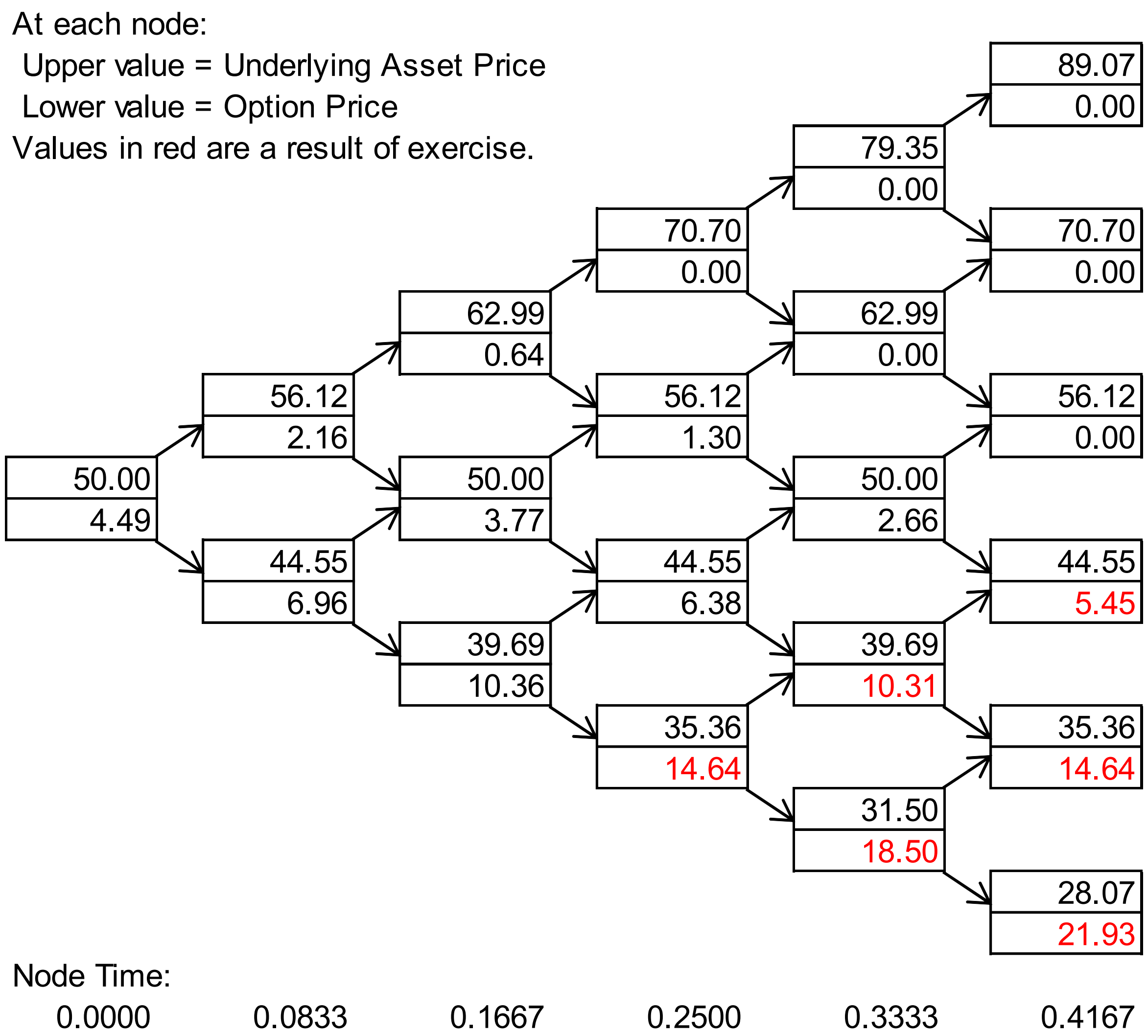
$$a = e^{0.1\times 1/12} = 1.0084$$

$$u = e^{0.4\sqrt{1/12}} = 1.1224$$

$$d = \frac{1}{u} = 0.8909$$

$$p = \frac{1.0084 - 0.8909}{1.1224 - 0.8909} = 0.5073$$

Backward iteration



Brownian Motion

What we cover

- Building the intuition and definition of Brownian motion
- Properties of Brownian motion
- Concept of Quadratic variation.

Symmetric Random Walks

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T, \end{cases}$$

$$M_k = \sum_{j=1}^{k} X_j, \; k = 1, 2, \ldots.$$

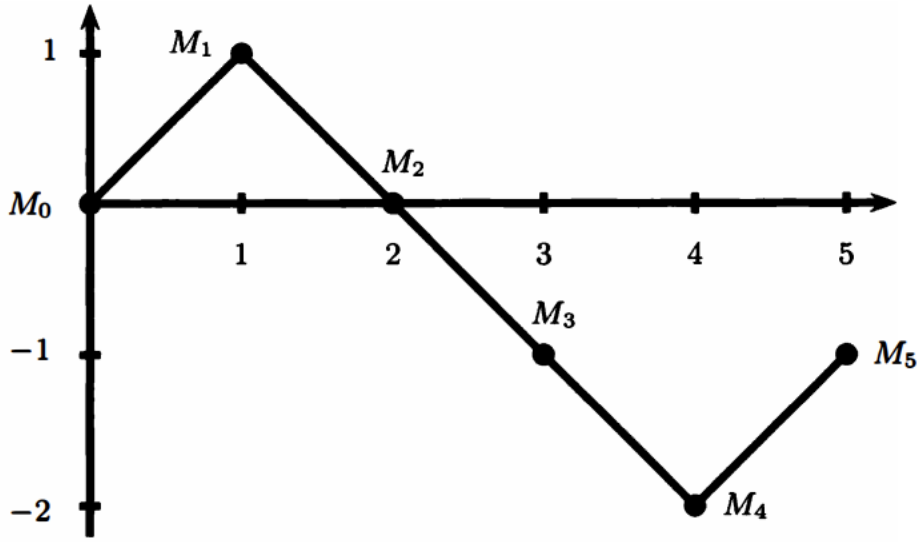


Fig. 3.2.1. Five steps of a random walk.

Properties of SRW

- The increments are independent

A random walk has *independent increments*. This means that if we choose nonnegative integers $0 = k_0 < k_1 < \cdots < k_m$, the random variables

$$M_{k_1} = (M_{k_1} - M_{k_0}),\ (M_{k_2} - M_{k_1}), \ldots, (M_{k_m} - M_{k_{m-1}})$$

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j,$$

- What is the expected value of each increment ?
- What is the variance of the increments ?

Martingale Property

- When k < l

$$\begin{aligned}\mathbb{E}\big[M_\ell \big| \mathcal{F}_k\big] &= \mathbb{E}\big[(M_\ell - M_k) + M_k \big| \mathcal{F}_k\big] \\ &= \mathbb{E}\big[M_\ell - M_k \big| \mathcal{F}_k\big] + \mathbb{E}\big[M_k \big| \mathcal{F}_k\big] \\ &= \mathbb{E}\big[M_\ell - M_k \big| \mathcal{F}_k\big] + M_k \\ &= \mathbb{E}\big[M_\ell - M_k\big] + M_k \;=\; M_k.\end{aligned}$$

Quadratic Variation of SRW

- The quadratic variation is defined along a path
- QV up to time k along a path is computed by taken all the one step increments of M along the path and squaring these increments

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k.$$

- What is the difference between the variance and QV ?

Scaled Symmetric Random Walk

- We want to speed up the coin toss and scale down the step up and down values of symmetric random walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt},$$

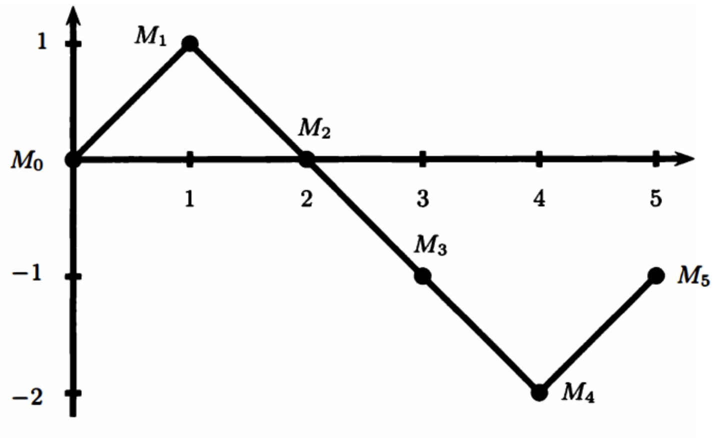


Fig. 3.2.1. Five steps of a random walk.

3.2 Scaled Random Walks 8

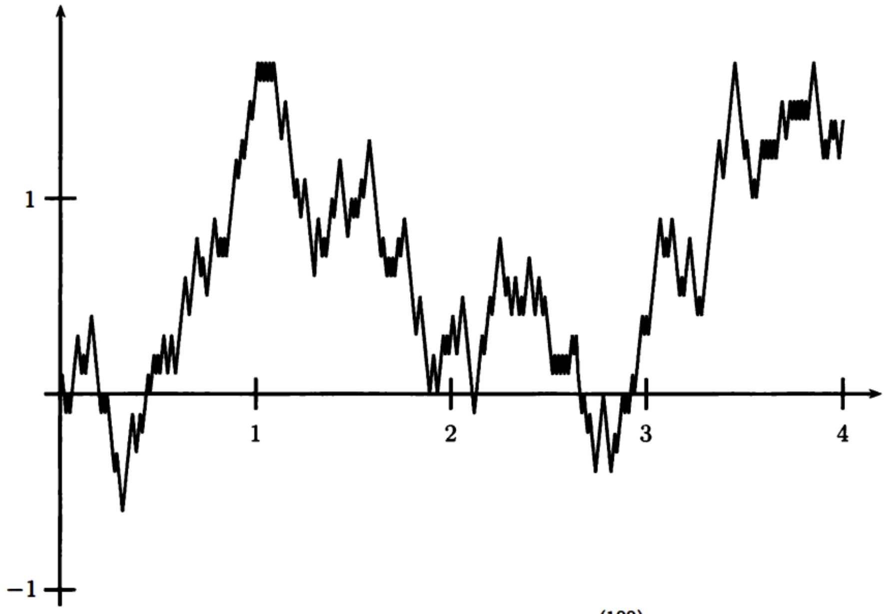


Fig. 3.2.2. A sample path of $W^{(100)}$.

Properties of SSRW

- The increments are independent

$$(W^{(n)}(t_1) - W^{(n)}(t_0)),\ (W^{(n)}(t_2) - W^{(n)}(t_1)), \ldots, (W^{(n)}(t_m) - W^{(n)}(t_{m-1}))$$

- The expectation of increments is:

$$\mathbb{E}\big(W^{(n)}(t) - W^{(n)}(s)\big) = 0,$$

- The variance of increments is

$$\mathrm{Var}\big(W^{(n)}(t) - W^{(n)}(s)\big) = t - s.$$

- SSRW is a martingale

$$\mathbb{E}\big[W^{(n)}(t) \big| \mathcal{F}(s)\big] = W^{(n)}(s)$$

Quadratic variation of SSRW

- The QV for a W(100) up till time 1.37 is

$$[W^{(100)}, W^{(100)}](1.37) = \sum_{j=1}^{137} \left[W^{(100)}\left(\frac{j}{100}\right) - W^{(100)}\left(\frac{j-1}{100}\right) \right]^2$$
$$= \sum_{j=1}^{137} \left[\frac{1}{10} X_j \right]^2 = \sum_{j=1}^{137} \frac{1}{100} = 1.37.$$

For any t

$$[W^{(n)}, W^{(n)}](t) = \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right) \right]^2$$
$$= \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j \right]^2 = \sum_{j=1}^{nt} \frac{1}{n} = t.$$

Brownian motion

- Limiting distribution of a SSRW is a Brownian Motion
- The figure shows the outcome of one series of coin tosses
- You can also fix time t and look at outcome of several series of coin tosses
- If t = 0.25, the possible values of $W^{(100)}(0.25)$?
- What is the probability of $W^{(100)}(0.25)=0.1$?

3.2 Scaled Random Walks

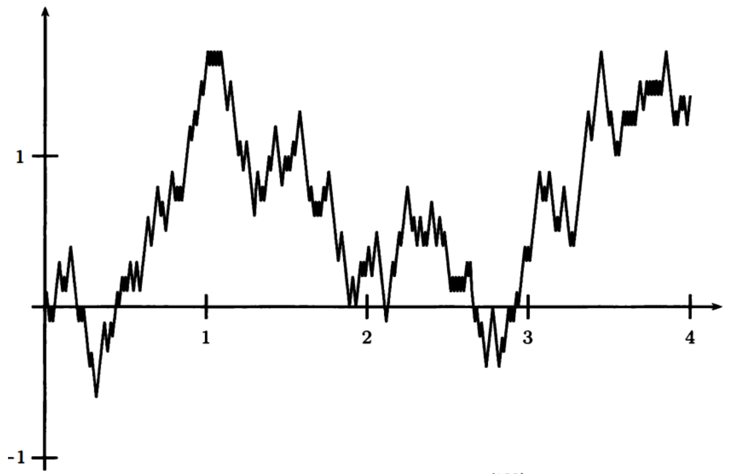
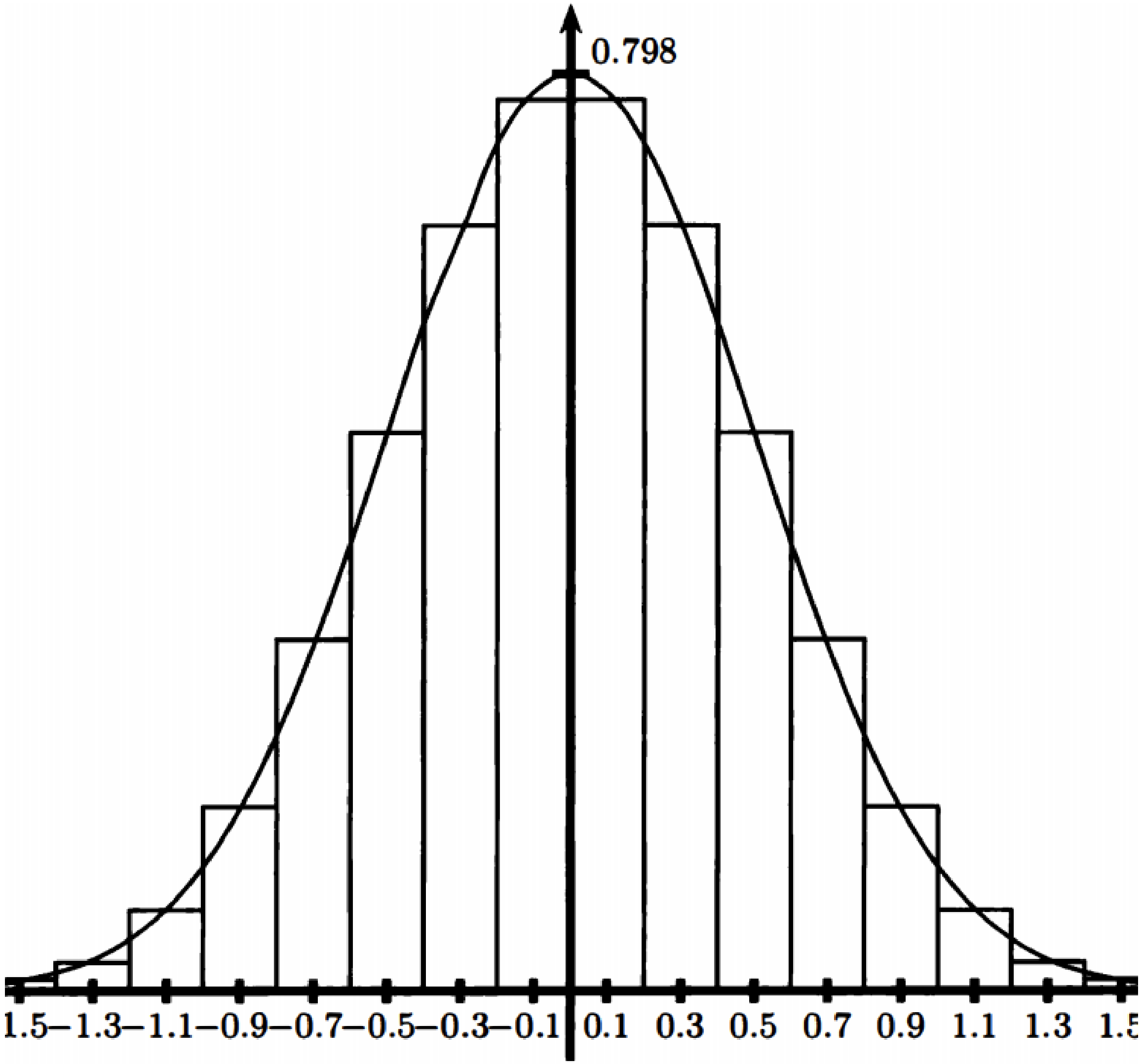


Fig. 3.2.2. A sample path of $W^{(100)}$.

Distribution of SSRW at t

- Figure shows the histogram bar centered at 0.1
- The width of the bar is 0.2, then what should be the height of the bar such that the probability of $W^{(100)}(0.25)=0.1555$?
- What is the variance of $W^{(100)}(0.25)$?



ig. 3.2.3. Distribution of $W^{(100)}(0.25)$ and normal curve $y = \frac{2}{\sqrt{2\pi}}e^{-2}$

Limiting distribution of SSRW

- For a fixed t, as n goes to infinity the distribution of scaled random walk converges to normal distribution, with mean ?, and variance ?
- PROOF
  - In order to identify a distribution one can identify their moment generating functions.
  - For normal density it would be

$$\begin{aligned}
\varphi(u) &= \int_{-\infty}^{\infty} e^{ux} f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{ ux - \frac{x^2}{2t} \right\} dx \\
&= e^{\frac{1}{2}u^2 t} \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(x-ut)^2}{2t} \right\} dx \\
&= e^{\frac{1}{2}u^2 t}
\end{aligned}$$

Proof continued

- Moment generating function for $W^{(n)}(t)$ will be

$$\varphi_n(u) = \mathbb{E}e^{uW^{(n)}(t)} = \mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}M_{nt}\right\}$$
$$= \mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}\sum_{j=1}^{nt}X_j\right\} = \mathbb{E}\prod_{j=1}^{nt}\exp\left\{\frac{u}{\sqrt{n}}X_j\right\}.$$

- The expectation can be taken inside, because of independence of X

$$\prod_{j=1}^{nt}\mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}X_j\right\} = \prod_{j=1}^{nt}\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right) = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)^{nt}.$$

- We need to show that as n goes to infinity the above converges to the moment generating function of normal distribution

Proof continued

- We need show that

$$\log \varphi_n(u) = nt \log\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)$$

Converges to $\log \varphi(u) = \frac{1}{2}u^2 t.$

- Take x = 1/sqrt(n)

$$\lim_{n\to\infty} \log \varphi_n(u) = t \lim_{x\downarrow 0} \frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}.$$

Apply L’ Hopital’s rule

- Numerator

$$\frac{\partial}{\partial x}\log\left(\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}\right)=\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}},$$

- Denominator is 2x
- Therefore,

$$\lim_{n\to\infty}\log\varphi_n(u)=t\lim_{x\downarrow 0}\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{2x\left(\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}\right)}=\frac{t}{2}\lim_{x\downarrow 0}\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{x},$$

Again apply L' Hopital's rule

- Numerator

$$\frac{\partial}{\partial x}\left(\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}\right) = \frac{u^2}{2}e^{ux} + \frac{u^2}{2}e^{-ux},$$

- Denominator is 1. Hence

$$\lim_{n\to\infty} \log \varphi_n(u) = \frac{t}{2}\lim_{x\downarrow 0}\left(\frac{u^2}{2}e^{ux} + \frac{u^2}{2}e^{-ux}\right) = \frac{1}{2}u^2 t,$$

Definition of Brownian Motion

- Brownian motion is obtained as the limiting case of scaled random walk

Definition 3.3.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. For each* $\omega \in \Omega$, *suppose there is a continuous function* $W(t)$ *of* $t \geq 0$ *that satisfies* $W(0) = 0$ *and that depends on* ω. *Then* $W(t)$, $t \geq 0$, *is a* Brownian motion *if for all* $0 = t_0 < t_1 < \cdots < t_m$ *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \ldots,\ W(t_m) - W(t_{m-1}) \quad (3.3.1)$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}\big[W(t_{i+1}) - W(t_i)\big] = 0, \quad (3.3.2)$$

$$\mathrm{Var}\big[W(t_{i+1}) - W(t_i)\big] = t_{i+1} - t_i. \quad (3.3.3)$$