Introduction to Portfolio Theory

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Session goals:

- Understand risk, return, and diversification concepts.
- Construct and interpret efficient frontier portfolios.
- Apply CAPM to evaluate systematic risk.

Session Agenda

- Risk & return
- Two-asset & multi-asset portfolios
- Diversification
- Efficient frontier
- CAPM and Beta

Motivation

- Investors rarely hold a single security
- Portfolio outcomes depend on:
  - Individual asset behavior
  - How assets move together
- Central idea: **Risk can be reduced without sacrificing expected return**
- This insight earned Markowitz the Nobel Prize.

Return as a Random Variable

- For a single asset with price $S(0),\ S(1)$, the return is:

$$K = \frac{S(1) - S(0)}{S(0)}$$

- Return is random because $S(1)$ is uncertain.
- Expected return measures central tendency:

$$\mu = \mathbb{E}(K)$$

- Risk is captured by standard deviation:

$$\sigma = \sqrt{\mathrm{Var}(K)}$$

Why Variance as Risk

- Captures spread of possible outcomes.
- Penalizes deviations both above and below the mean (risk-averse stance).
- Works naturally with linear combinations of assets.
- Integrates directly into optimization problems (quadratic structure).

Two Securities: Portfolio Return

For weights w_1, w_2 with $w_1 + w_2 = 1$:

- Portfolio return:

$$K_V = w_1 K_1 + w_2 K_2$$

- Portfolio expected return:

$$\mu_V = w_1 \mu_1 + w_2 \mu_2$$

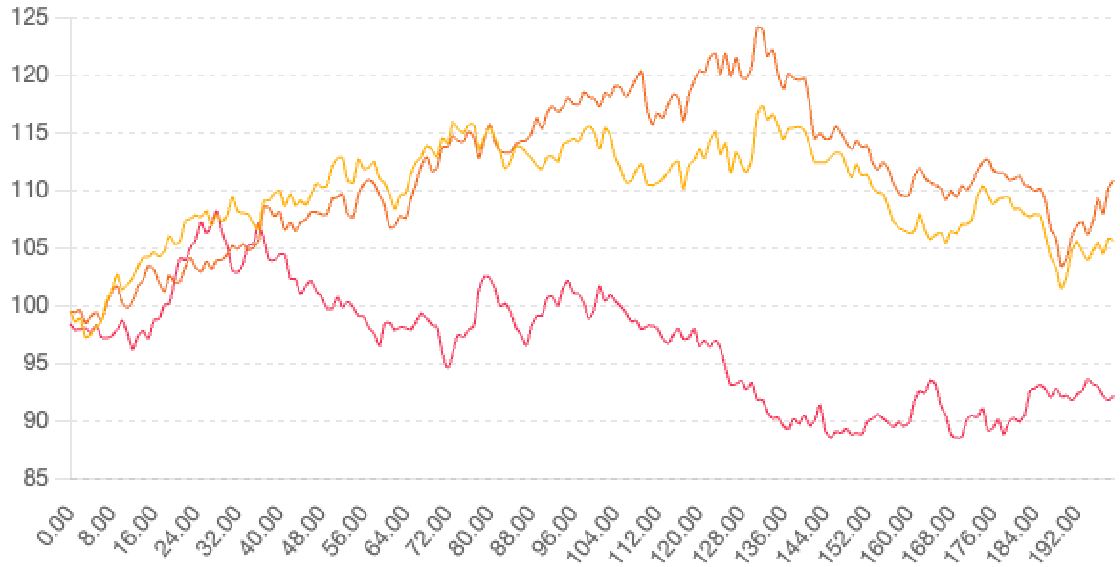
- This is a linear, intuitive relationship.

Two Securities: Portfolio Risk

- Portfolio variance:
  $$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\text{Cov}(K_1, K_2)$$

- Using correlation ρ_{12}:
  $$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- **Key insight:** Risk depends on *co-movement*, not individual volatilities alone.



Diversification Mechanism

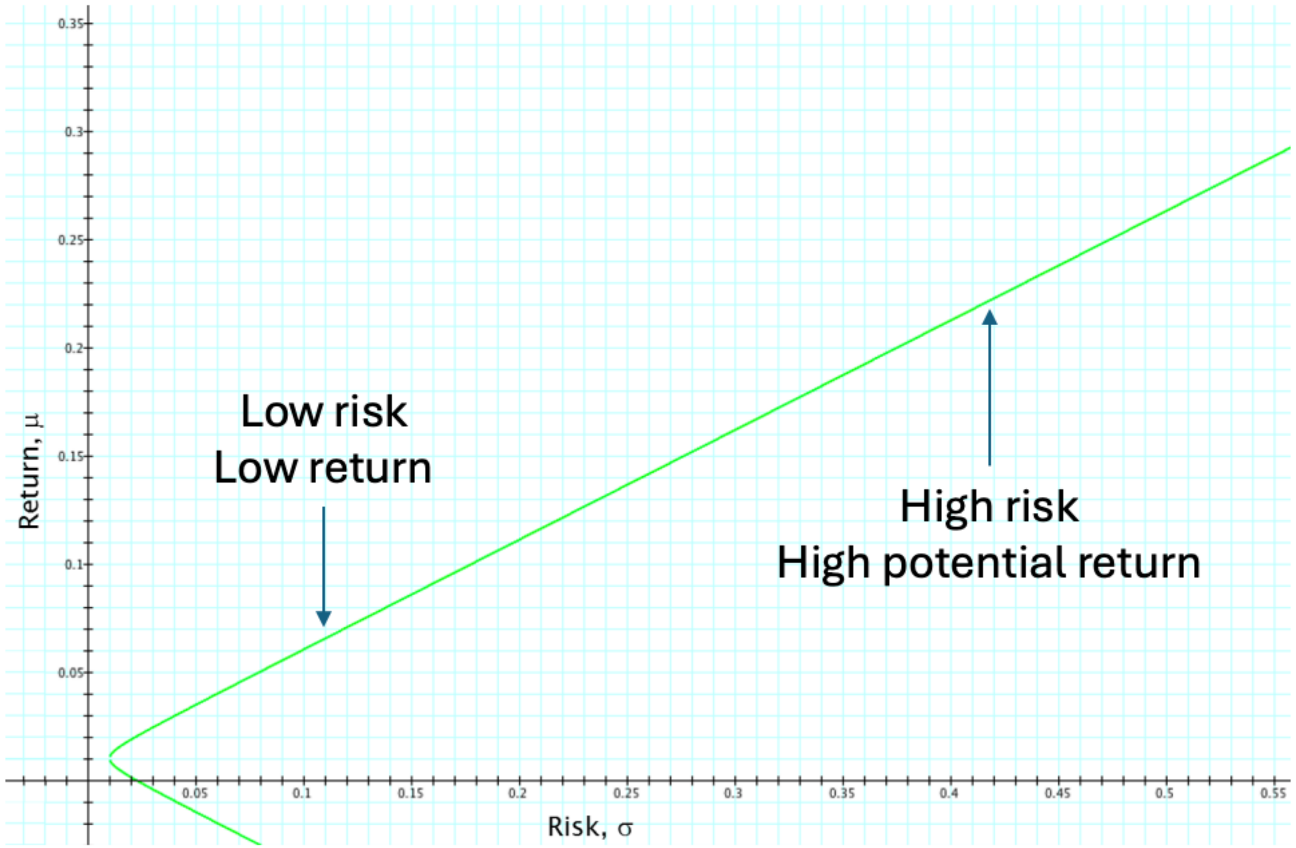
$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- If $\rho_{12} < 1$, diversification reduces portfolio variance.
- Greatest benefit when ρ_{12} is *negative*.
- Even when both assets are individually risky, the combination may be *less risky than either*.
- Diversification cannot eliminate *systematic* (market-wide) risk.

Two-asset Theory: Core Idea

Two-Asset Theory states:

- Any efficient portfolio composed of two risky assets must lie on a smooth curve in (σ, μ) space.
- The curve is parabolic when variance is plotted against weight.
- Every investor choosing between only two risky assets will pick a point on this curve based on risk preference.
- All efficient portfolios of two risky assets are convex combinations of the two assets.



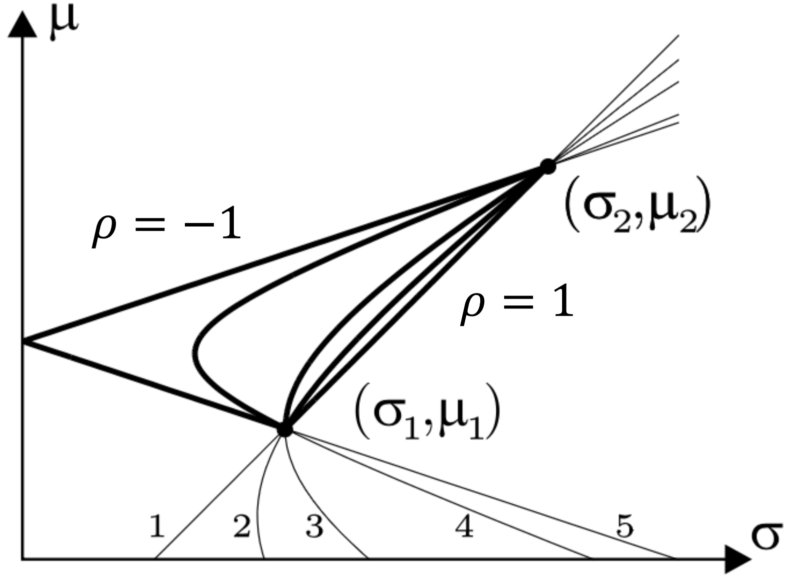
Two-asset Efficient Frontier

For assets 1 and 2, the efficient frontier is the **upper branch** of the curve generated by:

$$\mu(w) = w\mu_1 + (1-w)\mu_2$$

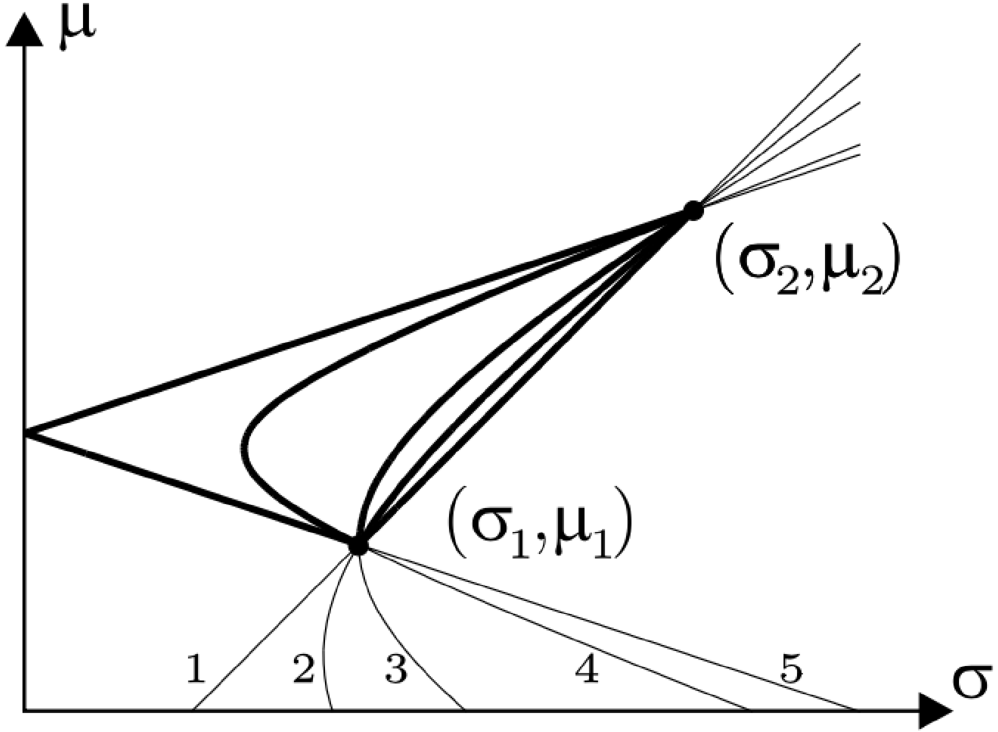
$$\sigma^2(w) = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2$$

- Each point corresponds to a w.
- Upper part of the curve is *efficient*.
- Lower part of the curve is dominated.
- Endpoints correspond to holding 100% of one asset.
- Diversification improves risk-return trade-offs.



Conditions for the Efficient Set

- If $\frac{\sigma_1}{\sigma_2} < \rho_{12} \leq 1$, then there is a portfolio with short selling such that $\sigma_V < \sigma_1$, but for each portfolio without short selling $\sigma_V \geq \sigma_1$ (lines 1 and 2).
- If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $\sigma_V \geq \sigma_1$ for each portfolio (line 3).
- If $-1 \leq \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then there is a portfolio without short selling such that $\sigma_V < \sigma_1$ (lines 4 and 5).



Minimum Variance Portfolio

For $-1 < \rho_{12} < 1$ the portfolio with minimum variance is attained at

$$w_0 = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

How?

$$\sigma_V^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2$$

- Step-1: Compute the derivative and equate it to 0.

$$\frac{d\sigma_V^2}{dw} = 2w\sigma_1^2 - 2(1-w)\sigma_2^2 + 2(1-w)\rho_{12}\sigma_1\sigma_2 - 2w\rho_{12}\sigma_1\sigma_2 = 0$$

- Step-2: Check if the second derivative is positive.

$$2\sigma_1^2 + 2\sigma_2^2 - 4\rho_{12}\sigma_1\sigma_2 > 2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_1\sigma_2 = 2(\sigma_1 - \sigma_2)^2 \geq 0$$

Portfolio with several securities

- A portfolio with n securities can be described in terms of its weights

$$w_i = \frac{x_i S_i(0)}{V(0)}, \qquad i = 1, \dots, n$$

where x_i is the number of shares of type i in the portfolio.

Matrix representation

$$\boldsymbol{w} = [w_1 \quad w_2 \quad \dots \quad w_n]$$

Their weights adding up to 1 can be written as:

$$1 = \boldsymbol{u}\boldsymbol{w}^T$$

where $\boldsymbol{u} = [1 \quad 1 \quad \dots \quad 1]$

The expected returns of the portfolio are represented as

$$\boldsymbol{m} = [\mu_1 \quad \mu_2 \quad \dots \quad \mu_n]$$

Portfolio with several securities

The covariance matrix is written as:

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

Hence, the expected return and variance of the portfolio can be given as:

$$\mu_V = \boldsymbol{m}\boldsymbol{w}^T; \;\; \sigma_V^2 = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T$$

- The diagonal elements are simply the variance of returns: $c_{ii} = \mathrm{Var}(K_i)$.
- $\boldsymbol{C}$ is a symmetric and positive definite matrix, where the inverse $\boldsymbol{C}^{-1}$ is possible.

Minimum Variance Portfolio

The portfolio with the smallest variance has weights:

$$\boldsymbol{w_0} = \frac{\boldsymbol{u}\boldsymbol{C}^{-1}}{\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T}$$

How?

$$F(\boldsymbol{w}, \lambda) = \boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T - \lambda(\boldsymbol{u}\boldsymbol{w}^T - 1)$$

Equating the partial derivative of F with w_i, we get

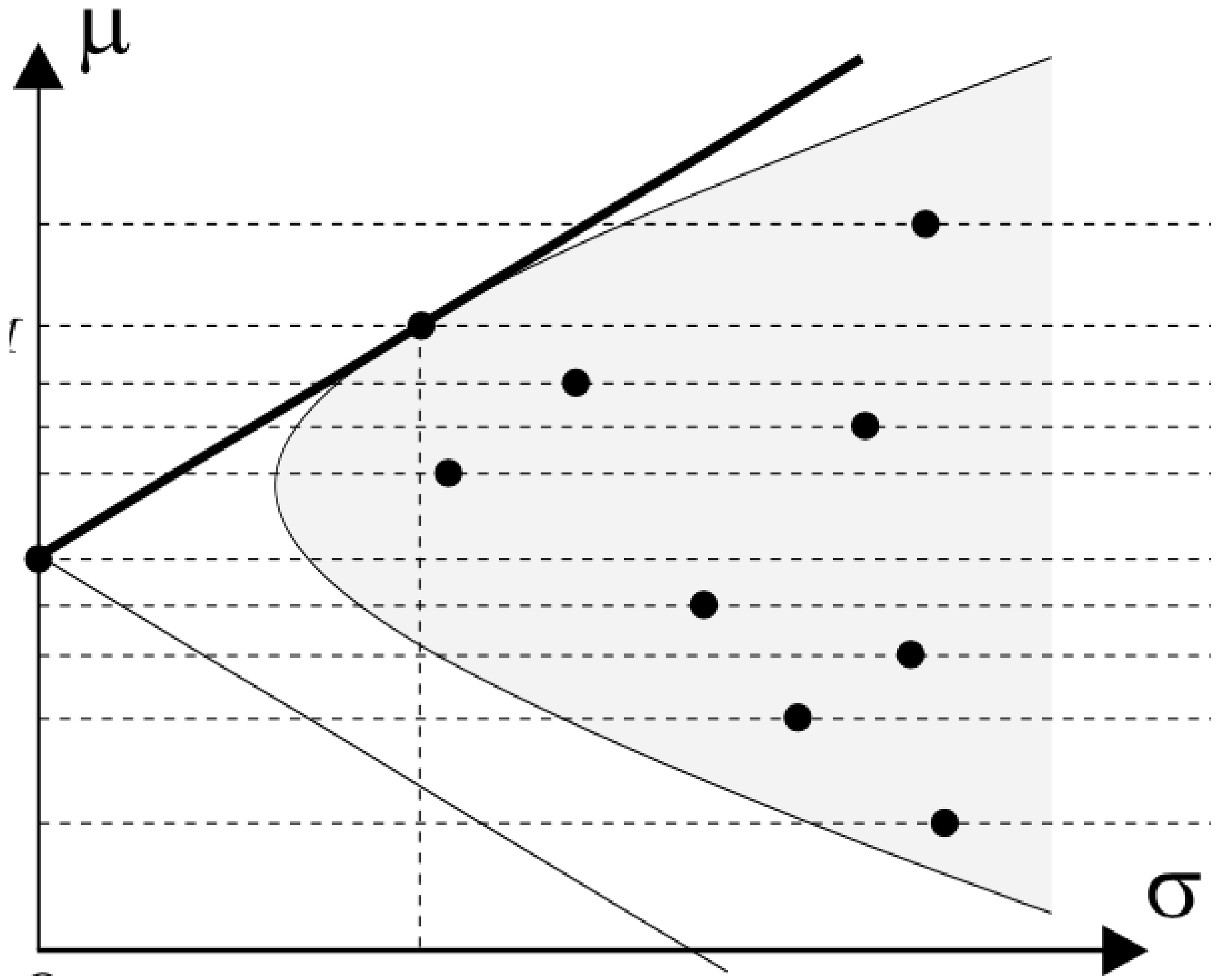
$$2\boldsymbol{w}\boldsymbol{C} - \lambda\boldsymbol{u} = 0 \Rightarrow \boldsymbol{w} = \frac{\lambda}{2}\boldsymbol{u}\boldsymbol{C}^{-1}$$

Substituting this in the constraint $1 = \boldsymbol{u}\boldsymbol{w}^T$, we get:

$$1 = \frac{\lambda}{2}\boldsymbol{u}\boldsymbol{C}^{-1}\boldsymbol{u}^T$$

Solving the above equations gives us $\boldsymbol{w_0}$.

Minimum Variance for an Expected Return



Minimum Variance for an Expected Return

Solve: $\min \boldsymbol{wCw}^T$, subject to: (a) $\boldsymbol{wm}^T = \mu$, and (b) $\boldsymbol{wu}^T = 1$

$$G(\boldsymbol{w}, \lambda_1, \lambda_2) = \boldsymbol{wCw}^T - \lambda_1(\boldsymbol{wm}^T - \mu) - \lambda_2(\boldsymbol{wu}^T - 1)$$

Taking partial derivatives with respect to $\boldsymbol{w}$, λ_1 and λ_2, and solving, we get.

$$2\boldsymbol{w} = \lambda_1 \boldsymbol{mC}^{-1} + \lambda_2 \boldsymbol{uC}^{-1}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

where,

$$M = \begin{bmatrix} \boldsymbol{mC}^{-1}\boldsymbol{m}^T & \boldsymbol{uC}^{-1}\boldsymbol{m}^T \\ \boldsymbol{mC}^{-1}\boldsymbol{u}^T & \boldsymbol{uC}^{-1}\boldsymbol{u}^T \end{bmatrix}$$

The final solution is of the form: $\boldsymbol{w} = \mu\boldsymbol{a} + \boldsymbol{b}$

Two-fund Theorem

Let w_1, w_2 be the weights of any two portfolios V_1, V_2 on the minimum variance line with expected returns $\mu_{V_1} \neq \mu_{V_2}$.

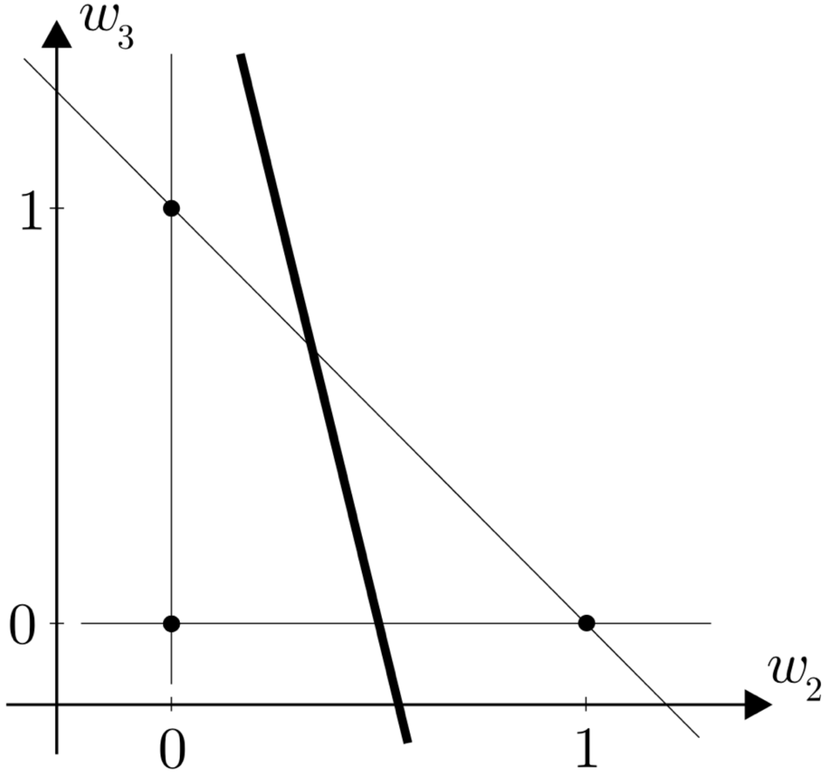
Each portfolio V on the minimum variance line can be expressed as:

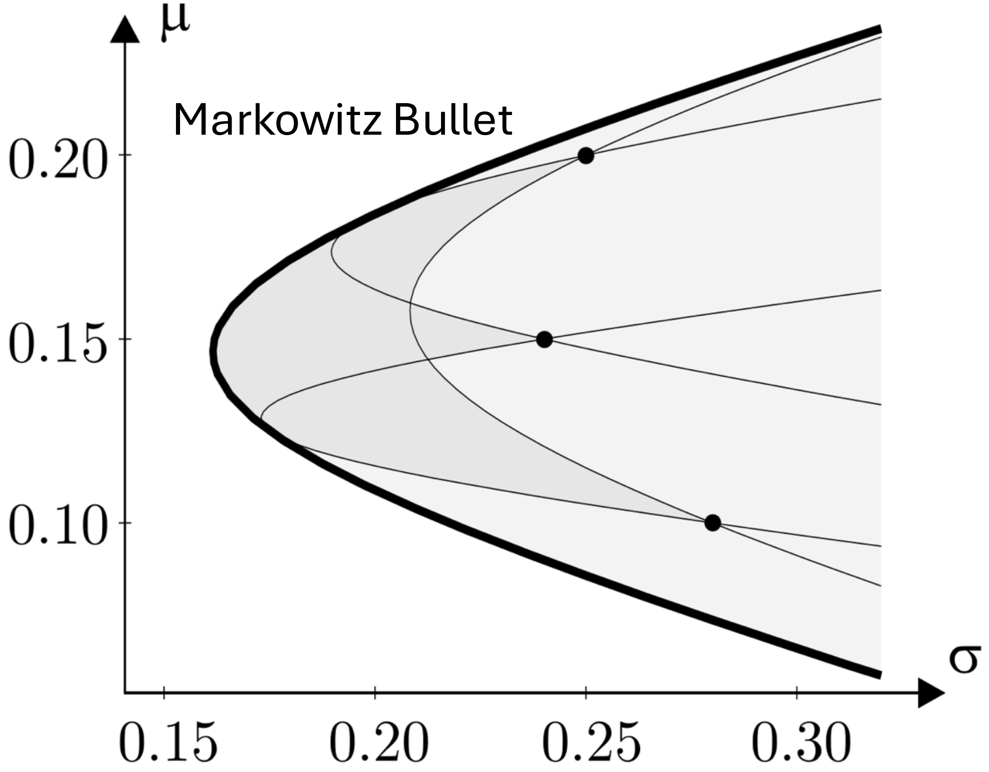
$$\boldsymbol{w} = \alpha \boldsymbol{w}_1 + (1 - \alpha)\boldsymbol{w}_2$$

Practical significance

- Any portfolio on the minimum variance line can be realized by splitting the wealth between just two investment funds.
- Trading the units of only two investment funds can significantly reduce the costs and simplify the procedure as compared to simultaneous transactions in n risky assets.
- Ofcourse the target expected returns can only be between μ_{V_1} and μ_{V_2}.

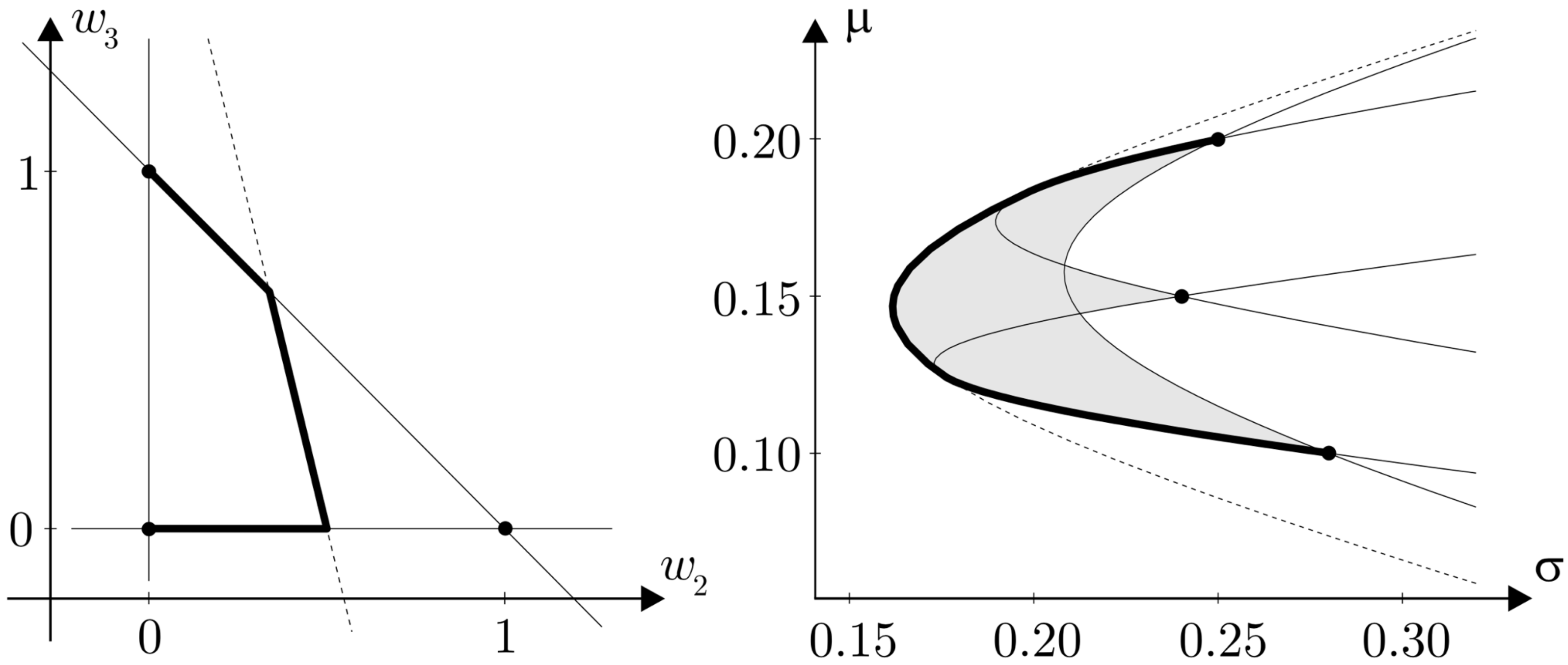
Feasible portfolio with 3 securities





Feasible portfolio with 3 securities

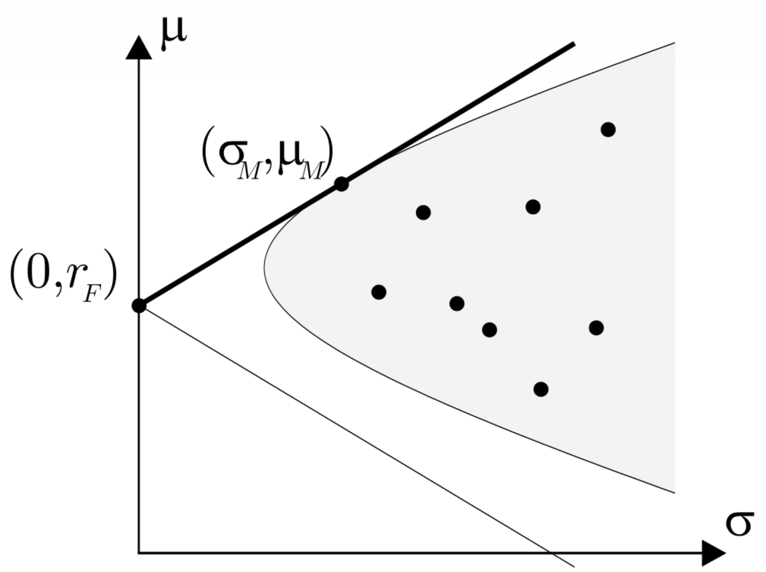
without short selling



Market Portfolio

- Let's add a risk-free security with return r_F.
- Every rational investor expecting a dominating portfolio will select a portfolio on this half-line, called the **Capital Market Line (CML).**
- The argument works until r_F is not too high, i.e., it is less than the expected return of the min. variance portfolio.
- (σ_M, μ_M) is referred to as the **Market Portfolio**.
- This is the steepest (highest gradient) line passing through $(0, r_F)$, given as:

$$\frac{\mu_V - r_F}{\sigma_V} = \frac{\boldsymbol{w}\boldsymbol{m}^T - r_F}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}}$$



Market Portfolio

- To calculate the market portfolio, we need to maximize the slope:

$$\max \frac{\boldsymbol{w}\boldsymbol{m}^T - r_F}{\sqrt{\boldsymbol{w}\boldsymbol{C}\boldsymbol{w}^T}}, \qquad \mathrm{s.t.}\, \boldsymbol{w}\boldsymbol{u}^T = 1$$

- Solving it through the Lagrange function, leads to

$$\boldsymbol{w}_M = \frac{(\boldsymbol{m} - r_F\boldsymbol{u})\boldsymbol{C}^{-1}}{(\boldsymbol{m} - r_F\boldsymbol{u})\boldsymbol{C}^{-1}\boldsymbol{u}^T}$$

- Any portfolio (σ, μ) on the CML satisfies: $\mu = r_F + \frac{\mu_M - r_F}{\sigma_M}\sigma$
  - The term $\frac{\mu_M - r_F}{\sigma_M}\sigma$ is called the *risk premium*.
  - It is the additional return over and above the risk-free return r_F, which compensates for exposure to risk.

CAPM: Capital Asset Pricing Model

- We know that CML is tangent to the efficient frontier at (σ_M, μ_M).
- Assume another portfolio V characterized by (σ_V, μ_V).
- We can construct a range of portfolios from M and V.
- These portfolios should form a hyperbola which should technically be a tangent to the CML at (σ_M, μ_M).
- Assume the weight of any such portfolio to be x for V and $(1 - x)$ for M.

$$\sigma_P = \left(x^2\sigma_V^2 + (1-x)^2\sigma_M^2 + 2x(1-x)\mathrm{Cov}(K_V, K_M)\right)^{\frac{1}{2}}$$

$$\mu_P = x\mu_V + (1-x)\mu_M$$

CAPM: Capital Asset Pricing Model

- Taking derivatives with respect to x at $x = 0$, we get

$$\left.\frac{\partial \sigma_P}{\partial x}\right|_{x=0} = \frac{\mathrm{Cov}(K_V, K_M) - \sigma_M^2}{\sigma_M}; \left.\frac{\partial \mu_P}{\partial x}\right|_{x=0} = \mu_V - \mu_M$$

- The slope of CML should equate the slope of the curve at $x = 0$.

$$\frac{\mu_V - \mu_M}{\frac{\mathrm{Cov}(K_V, K_M) - \sigma_M^2}{\sigma_M}} = \frac{\mu_M - r_F}{\sigma_M}$$

- Solving this gets us:

$$\mu_V = r_F + \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}(\mu_M - r_F)$$

- This is referred to as the *beta factor,* given as: $\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}$.

CAPM: Capital Asset Pricing Model

CAPM theorem

- *Suppose that the risk-free rate r_F is lower than the expected return μ_{MVP} of the minimum variance portfolio (so that the market portfolio M exists).*
- *Then, the expected return μ_V on any feasible portfolio V is given by*
$$\mu_V = r_F + \beta_V(\mu_M - r_F)$$
- *The term $\beta_V(\mu_M - r_F)$ is called risk premium.*
- *This becomes same as the risk premium of the market portfolio if V is on the capital market line.*
$$\beta_V(\mu_M - r_F) = \frac{\mu_M - r_F}{\sigma_M}\sigma_V$$
- *Suppose we want to approximate K_V by a linear function $\beta K_M + \alpha$, the error of approximation is given as*
$$\epsilon = K_V - (\beta K_M + \alpha)$$

Thank you!