POLE-ZERO DECOMPOSITION :

A NEW TECHNIQUE FOR DESIGN OF DIGITAL FILTERS

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ABSTRACT

A new technique for design of digital filters is presented in this paper. The technique consists of splitting the given log magnitude response into two parts, one corresponding to the response of the numerator polynomial of the filter transfer function and the other part to the response of the denominator polynomial. The inverse of each of these polynomials is considered as an all-pole filter and the response of the all-pole filter is approximated by a small number of autoregressive coefficients. The autoregressive coefficients obtained for the numerator polynomial represent the zero part of the final filter and the coefficients obtained for the denominator polynomial represent the pole part of the final filter. With equal number of poles and zeros, the overall filter response can be made nearly equiripple in the passband and stopband. The amplitude of the ripple can be traded with the width of the transition band. The ripple characterstics can be controlled by appropriately choosing the number of poles and zeros of the filter.

INTRODUCTION

A digital filter that contains poles and zeros is termed as an autoregressive and moving average (ARMA) digital filter. This paper is concerned with the design of ARMA digital filters to realize a given log magnitude frequency response. The basic idea in the design is to split the given response into two component responses, each of which can be approximated by a small number of parameters. The reason for the success of this method is that one of the component responses is close to an all-pole spectrum and the other component response is close to an all-zero spectrum. Since the inverse of an all-zero spectrum is an all-pole spectrum, it is possible to represent each of the component responses by a small number of parameters through autoregressive modelling [1]. The splitting of the given response into an all-pole and an all-zero spectra is accomplished using a pole-zero decomposition technique [2], which is based on the properties of the negative derivative of minimum phase spectra [3].

It is generally true that a given magnitude spectrum can be realized by a digital filter of a much lower order when the filter contains both poles and zeros than when the filter is purely all-pole or all-zero. A lower order digital filter will be of lesser complexity in terms of number of multiplications and additions required in its implementation. Our method of design results in an ARMA digital filter that is of low order and stable. In addition, the filter has several useful characteristics. The ripple chracteristics can be controlled by a suitable choice of the number of coefficients used to represent the component spectra. The ampiltude of the ripple can be traded with the width of the transition band. Since all the poles and zeros of the filter lie within the unit circle in the z-plane, the inverse filter will have the complement response and will be stable.

The emphasis in this paper is on the presentation of the new design technique. Issues such as the filter performance relative to other techniques and the limitations of the method are not considered here. Throughout the paper the notation (M1,M2) denotes an ARMA digital filter with M1 pole and M2 zeros.

DESIGN PROCEDURE

Pole-zero Decomposition

The key idea in this paper is splitting the given log magnitude frequency response into two parts, one corresponding nearly to an all-pole filter spectrum and the other to an all-zero filter spectrum. This splitting is called pole-zero decomposition, which is based on the properties of the negative derivative of phase spectra (NDPS) of minimum phase polynomials.

A linear digital filter H(z) can be represented as a ratio of two polynomials as follows:

$$H(z) = G\, N(z) / D(z) \qquad (1)$$

where

$$N(z) = 1 + \sum_{k=1}^{M1} a^-(k)\, z^{-k},$$

$$D(z) = 1 + \sum_{k=1}^{M2} a^+(k)\, z^{-k},$$

and G is a gain term.

The roots of the numerator polynomial are called zeros and the roots of the denominator polynomial are called poles. The objective in filter design is to determine the coefficients $\{a^-(k)\}$ and $\{a^+(k)\}$ such that the magnitude-squared frequency response of H(z) matches the given magnitude response as closely as possible. In this paper we refer to N(z) as zero part and D(z) as pole part of the digital filter.

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A polynomial is said to be minimum phase if all its roots lie within the unit circle in the z-plane. If all the zeros and poles of H(z) lie within the unit circle in the z-plane, then the filter is called a minimum phase filter. Using the properties of the negative derivative of minimum phase filters [2] it is possible to separate the significant contributions of poles and zeros in the combined NDPS response of a pole-zero filter by considering the positive and negative portions respectively.

Filter Design

Given a magnitude-squared frequency response S(w), the objective in filter design is to determine the parameters of a linear system model H(z) as given in (1) such that

$$| H(w) |^2 = | H(z) H^*(z) |_{z=e^{jw}} \simeq S(w).$$

Let V(w) be the Fourier transform of the minimum phase correspondent for S(w). Then

$$| V(w) |^2 = S(w).$$

The steps in the design are as follows:

1. Find the cepstral coefficients {c(k)} of S(w) using the relation

$$\ln | V(w) |^2 = \ln S(w) = c(0) + 2 \sum_{k=1}^{\infty} c(k) \cos kw .$$

2. Compute the NDPS $\theta_V'(w)$ from {c(k)} using the relation

$$\theta_V'(w) = \sum_{k=1}^{\infty} k\, c(k) \cos kw$$

3. Split $\theta_V'(w)$ into positive and negative portions.

$$\theta_V'(w) = [\theta_V'(w)]^+ + [\theta_V'(w)]^-,$$

where

$$[\theta_V'(w)]^+ = \theta_V'(w) \text{ , for } \theta_V'(w) > 0 \text{ ,}$$
$$= 0 \text{ , for } \theta_V'(w) \le 0 \text{ ,}$$

and

$$[\theta_V'(w)]^- = \theta_V'(w) \text{ , for } \theta_V'(w) \le 0 \text{ ,}$$
$$= 0 \text{ , for } \theta_V'(w) > 0 \text{ .}$$

4. Find the cepstral coefficients $\{c^+(k)\}$ and $\{c^-(k)\}$ from $[\theta_V'(w)]^+$ and $[\theta_V'(w)]^-$ respectively using the Fourier series expansions,

$$[\theta_V'(w)]^+ = C + \sum_{k=1}^{\infty} k\, c^+(k) \cos kw$$

and

$$[\theta_V'(w)]^- = -C + \sum_{k=1}^{\infty} k\, c^-(k) \cos kw \text{ ,}$$

where C is the average value, which does not contribute to the shape of the spectrum.

5. Compute the pole spectrum P(w) and the zero spectrum Z(w) from $\{c^+(k)\}$ and $\{c^-(k)\}$ respectively as follows:

$$\ln P(w) = c(0)/2 + 2 \sum_{k=1}^{\infty} c^+(k) \cos kw$$

and

$$\ln Z(w) = c(0)/2 + 2 \sum_{k=1}^{\infty} c^-(k) \cos kw.$$

Note that $c^+(k) + c^-(k) = c(k)$ and hence $P(w)\, Z(w) = S(w)$.

6. Find the autocorrelation coefficients $R^+(k)$ and $R^-(k)$ from P(w) and 1/Z(w) respectively using the relations

$$P(w) = R^+(0) + 2 \sum_{k=1}^{\infty} R^+(k) \cos kw$$

and

$$1/Z(w) = R^-(0) + 2 \sum_{k=1}^{\infty} R^-(k) \cos kw.$$

7. Solve for the autoregressive coefficients $\{a^+(k)\}$ and $\{a^-(k)\}$ from $\{R^+(k)\}$ and $\{R^-(k)\}$ respectively using Levinson's algorithm for solving the autocorrelation normal equations [1].

8. Compute the approximate pole spectrum $\hat{P}(w)$ from $\{a^+(k)\}$ and the approximate zero spectrum Z(w) from $\{a^-(k)\}$

9. The overall filter response is given by $| H(w) |^2 = \hat{P}(w)\, \hat{Z}(w)$.

10. The filter H(z) is given by (1). The values of M1 and M2 determine the order of the filter.

DESIGN EXAMPLES

A lowpass filter with the following specifications is considered for illustrating the above design procedure.

Let $S(l) = S(w) |_{w=2\pi l/512}$, $l = 0, 1, \ldots, 511$.

A = amplitude, and M = number of transition samples.

Specification:

$$\ln[S(l)] = \ln(A) \text{ ,} \quad l = 0, 1, \ldots 99$$
$$= [1-(l-99)/(M+1)] \ln(A), \quad l = 100, 101, \ldots 100+M$$
$$= 0 \text{ ,} \quad l = 100+M+1, 100+M+2, \ldots 256$$
$$\ln [S(l)] = \ln [S(512-l)], \quad l = 257, 258, \ldots 511.$$

In the above specification the value of A determines the level of stopband rejection. For example, if A=1000, then the stopband rejection level is 30 dB. The value of M determines the number of transition samples. The case of M=0 corresponds to no sample in the transition band. The filter design was carried out using 512 point FFT for computing Fourier transforms.

Fig. 1 illustrates the principle of the proposed filter design. Fig. 1a shows the log magnitude response of the desired lowpass filter for A=1000, M=5. The NDPS of the filter is shown in Fig. 1b. The positive and negative portions of the NDPS are separated and the corresponding spectra are computed. The resulting pole and zero spectra are shown in Fig. 1c. If these log spectra are added, we get the desired filter response exactly as shown in Fig. 1d. On the other hand, if the pole spectrum and the inverse of the zero spectrum are approximated by autoregressive models, each with 8 coefficients, the overall response of the resulting filter is as shown in Fig. 1e. In this case the peak to peak amplitude of the ripple is less than 8% in passband and stopband.

That the amplitude of the ripple can be traded with either the width of the transition band or the complexity (order) of the filter, is illustrated in Fig. 2. The filter responses for four different orders of the filter and three different transition widths are given in the figure. For a (16,16) filter for example, the ripple amplitude reduces from 11% for M=1 to 4% for M=11. This trade-off characteristic of the ripple amplitude with transition width makes this design somewhat superior to the statistical design of ARMA digital filters reported in [4].

The effect of varying the number of of zeros keeping the number of poles fixed is shown in Fig. 3. The passband characteristics are not significantly affected by changing the number of zero coefficients. Similarly, we observed that the stopband characteristics are not significantly affected by changing the number of pole coefficients. This will provide the flexibilty to design a filter with any desired passband and stopband characteristics.

Finally the design of a bandpass filter is illustrated in Fig. 4. This shows that any arbitrary filter characteristics can be realized using the technique presented in this paper.

CONCLUSIONS

We have shown that pole-zero decomposition technique provides an effective method for desgning ARMA digital filters. The complexity of the filter can be traded with the width of the transition band. By varying the number of poles and zeros independently any desired passband and stopband charateristics can be achieved. Since the roots of the resulting numerator and denominator polynomials lie within the unit cicle in the z-plane, the filter and its inverse both are gauranteed to be stable. Therefore, if H(z) represents a lowpass filter then 1/H(z) represnets a high pass filter. Similar reasoning applies for bandpass and and band elimination filters also.

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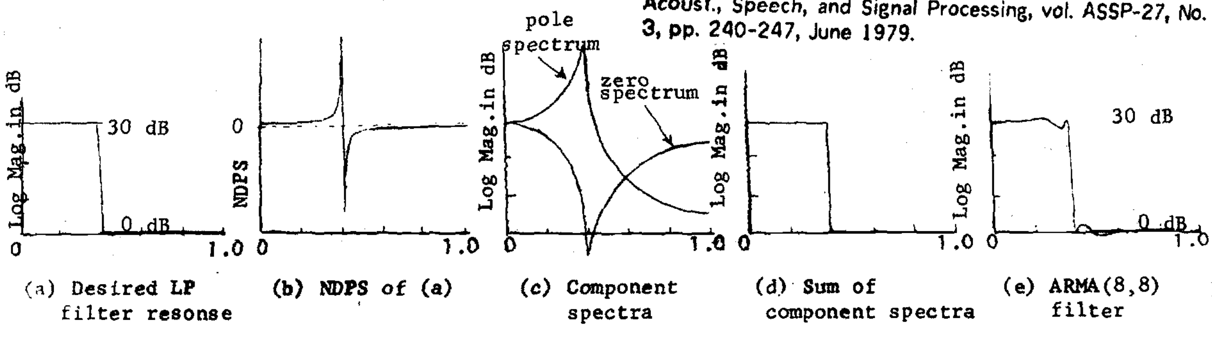

Fig. 1. Design of lowpass filter with a transition width of 5 samples

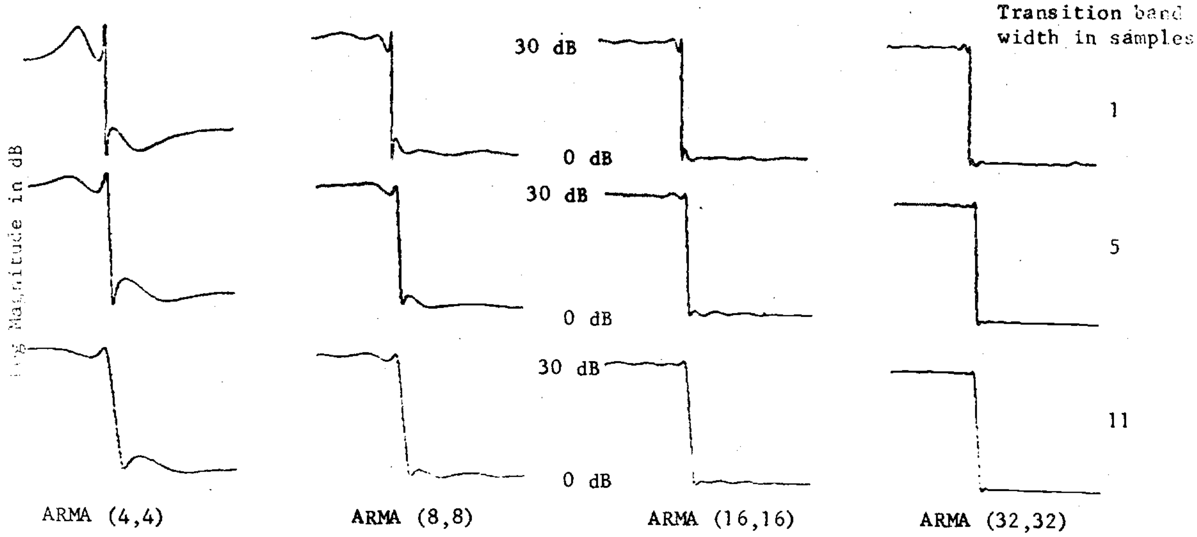

Fig. 2. Trade-off between ripple amplitude and width of the transition band for different orders of the filter.

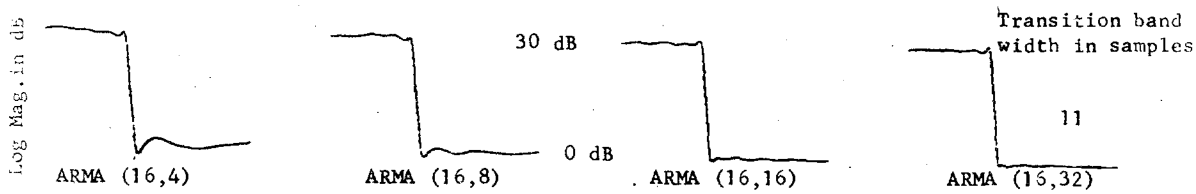

Fig. 3. Effect of varying the number of zeros of the ARMA digital filter.

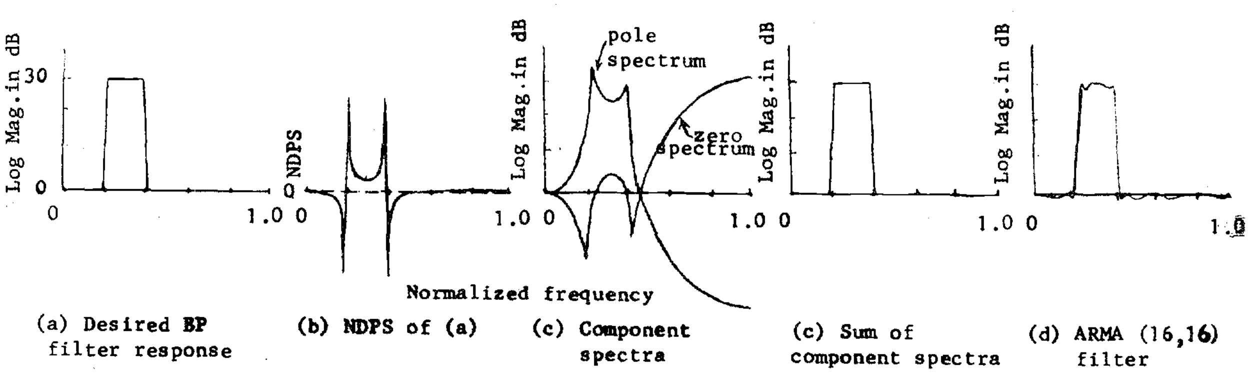

Fig. 4. Design of bandpass filter with a transition width of 5 samples