AN ALGORITHM FOR BANDLIMITED SIGNAL INTERPOLATION

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ABSTRACT

In this paper we present an algorithm for band-limited signal interpolation assuming the samples to be known at some randomly distributed instants. The algorithm is a modification of the Papoulis-Gerchberg algorithm. The key idea in our algorithm is to use some nonzero values at the missing points. The values are obtained using an interpolation scheme based on relaxation method for constraint propagation. The algorithm is illustrated with several examples.

INTRODUCTION

In this paper we consider the problem of interpolation of a band-limited signal from a set of values known only at a few sampling instants in a given interval. We propose an iterative algorithm to determine the missing samples. The main feature of the algorithm is that it uses a preliminary interpolation instead of setting the missing samples to zero. This feature improves significantly the convergence of the iterative solution. We also show that in many cases, it is not necessary to assume the location of the frequency band a priori, but it can be derived from the given data itself.

The problem of estimating the values of the missing samples from partial data arises in several practical situations. A good interpolation scheme should take into account not only the given data but also other available information such as the spectral content. An iterative technique was developed recently for recovery of missing samples from an over-sampled band-limited signal [1],[2]. This technique was based on Papoulis-Gerchberg algorithm, which was originally proposed for the extrapolation of a band-limited signal [3]. In this technique the missing samples of the time series are initially set to zero and the signal bandwidth is assumed a priori. During every iteration the Fourier Transform (FT) outside the given bandwidth is set to zero and the inverse transform is performed. In the time domain the samples at the known locations are replaced by the original data.

In this paper we show that a faster convergence to the solution can be obtained if some initial interpolation is done for the missing samples, instead of setting them to zero. Ripple effects, due to rectangular window in the frequency domain, can be reduced using a window with a smoother roll-off. Moreover, the location of the frequency band need not always be assumed a priori. It can be estimated from the available data itself in many cases.

BAND-LIMITED SIGNAL INTERPOLATION PROBLEM

We illustrate the band-limited signal interpolation problem through an example. Fig.1a shows a synthetic signal consisting of 256 samples of data and Fig.2a shows 10% of the samples at randomly selected instants. The objective is to determine the complete sample set of Fig.1a from the incomplete data shown in Fig.2a using the information that the signal is band-limited. In the frequency domain the desired spectral transformation is from Fig.2b to Fig.1b.

Previous methods attempt to achieve this by applying the Papoulis-Gerchberg iterative algorithm which involves the following steps:

1. Initially set the missing samples to zero in the given data.
2. Compute the Discrete Fourier Transform (DFT)
3. Band-limit the Fourier transform. That is, set the Fourier Transform outside the specified band to zero.
4. Compute the inverse DFT and replace the samples at the known instants with the sample values given in the original data.
5. Repeat steps 2 through 4 for a specified number of iterations or until some specified level of reconstruction is reached.

A close observation of the time and frequency domain operations in the above procedure suggests that there is scope to improve the rate of convergence with iterations or equivalently, to obtain a

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better reconstruction with fewer iterations. To achieve this improvement, we examine the following issues in our proposed method.

1. Use initially some interpolated values for the missing samples.
2. Derive the location and width information of the frequency band from the given partial data and from the data obtained after each iteration.
3 For band-limiting in the frequency domain, use a window with a smooth roll-off instead of a rectangular window.

In the following sections we discuss in detail the proposed modifications to the iterative interpolation algorithm.

PROPOSED METHOD FOR BAND-LIMITED SIGNAL INTERPOLATION

In the band-limited interpolation problem, the general tendency appears to be to set all the missing samples to zero. It is obvious that zero need not be the best choice. The problem of initial interpolation is to assign some values to the missing samples based on the given data and other available information. Any arbitrary way of interpolating at the unknown points will give rise to biased estimates. Ideally, we would like an estimate which is not only consistent with all the available information, but is also maximally noncommittal with respect to the unknown data. Similar ideas have been successfully used in the Maximum Entropy Method (MEM) for power spectrum estimation. Since a neat analytic approach like MEM is not obvious for the present problem, we develop an empirical approach to initial interpolation based on some heuristic arguments. We propose that the interpolated value should be made up of two components, one dependent on the given data and the other is a noise component that is added to increase the uncertainty. We now give a method to derive each of these components.

At present we have no convenient way of determining the data dependent component for all types of signals. But if we assume that the signal is band-limited in the low frequency region, there are many techniques available for interpolation. We have examined some of the standard interpolation techniques and also a relaxation procedure which was proposed as a constraint propagation algorithm [4]. We found that the relaxation procedure performs well compared to the standard interpolation techniques. This procedure is described through the equation

$$x_i(n)=C_i x_{i-1}(n)+\frac{1-C_i}{2}\left[x_i(n-1)+x_{i-1}(n+1)\right] \quad (1)$$

where

$x_i(n)$ = Interpolated value at the end of the i^{th} iteration

$C_i=i/N$ (Confidence factor)

N = Number of iterations

The interpolated values obtained using the above relaxation procedure for 20 iterations is shown in Fig.3a. The corresponding spectrum is shown in Fig.3b. Comparing this with the spectrum of the given data (Fig.2b), we notice that the signal peaks stand out much better in Fig.3b than in Fig.2b. At the same time, there is a significant bias towards the low frequency region in Fig.3b.

Uniformly distributed random numbers are weighted appropriately to generate samples of random noise to be added at the missing points. The weighting is done to account for the proximity of the missing sample, at, say, instant n, to the known samples at, say, instants n_1 and n_2, lying on either side of it. The additive noise is therefore generated using the formula

$$y(n)=\frac{4\,\bar{y}\,(n_2-n)(n-n_1)}{(n_2-n_1)^2}, \text{ for } n_1\le n\le n_2 \quad (2)$$

where

$Y(n)$ = Noise value at the sample index n

$\bar{y}$ = A random number uniformly distributed between the maximum and minimum of the known samples.

The two components, one due to data and the other due to noise are combined to obtain the initial interpolation samples.

$$z(n)=C_1\, x(n)+C_2\, y(n) \quad (3)$$

The values of the weighting constants C_1 and C_2 have to be determined empirically or experimentally.

Fig.4a shows the result of addition of noise to the relaxation interpolated values of Fig.3a. The spectrum of Fig.4a, shown in Fig.4b, clearly illustrates that the effect of additive noise is to reduce the low frequency bias, retaining at the same time the advantage of relaxation interpolation, namely, the signal peak stands out, unlike in the spectrum (shown in Fig.2b) of the given data.

We present a simple technique of extending the relaxation-based interpolation scheme to the high frequency signals. Given the knowledge of the band location, the center frequency f_o can be found. Then the given data is demodulated by multiplying with $\cos\omega_o n$. The effect of this multiplication is to translate the spectrum to a low frequency region. Now the relaxation interpolation can be performed on the demodulated sequence to obtain an estimate of the missing samples. The interpolated signal is remodulated by multiplying with $\cos\omega_o n$ again to bring the signal back to the original spectral range. The resulting signal is an initial interpolated signal which doesnot have a low frequency bias.

A striking feature of the band-limited signal interpolation problem is that even with as many as 90% of the samples missing, the frequency spectrum of the incomplete data set (with missing samples set to zero) exhibits in most cases the strongest peak around the band of interest. In such cases it is possible to locate the approximate region of the frequency band of the signal to start with, and refine it with each iteration as more information about the missing samples is gathered. Although we found this procedure to be working well in most cases, we have assumed that the bandwidth information is known for all the illustrations in this paper.

In order to evaluate the performance of our iterative procedure, we adopt the following error criterion. The total squared error of the reconstructed samples from the original samples is given by

$$E_o = \left[x(n) - x_o(n)\right]^2 \qquad (4)$$

where $x(n)$ and $x_o(n)$ are the reconstructed and original samples respectively. Since we are using simulated data for our studies, we know the complete signal $x_o(n)$, from which the partial data was generated.

RESULTS AND DISCUSSION

In this section we illustrate our iterative signal iterpolation method through simulated examples. We consider a band-limited signal of the type

$$s(n) = a_1 \sin(2\pi f_1 n T_S + \theta_1) + a_2 \sin(2\pi f_2 n T_s + \theta_2) \qquad (5)$$

where

T_S=sampling interval =1/512 seconds,

$a_1=1.25$, $\theta_1=\pi/3$ and $a_2=1.50$, $\theta_2=\pi/2$.

Since T_s is fixed, by varying f_1 and f_2 we can study both low frequency band high frequency cases. The partial data was generated by multiplying the complete signal s(n) with a random sequence of ones and zeros. Different distributions of samples can be obtained by varying the percentage of the orignal samples as well as their placement in the data.

In the illustrations to follow, we study mainly the effect of the proposed initial interpolation on the rate of convergence of the data to the original signal. The bandwidth of the signal is assumed to be known. A rectangular window is used in the frequency domain for band-limiting. The following parameter values are used in our illustrations.

value of C_1 in eq.(3): 1.0
value of C_2 in eq.(3): 1.0
Window in the frequency domain: f_o ±5 samples
$f_1 = 10$, $f_2 = 12$, $f_o = 11$

The results are presented as a plot of the mean squared error in the reconstructed signal as a function of the number of iterations. Fig.5a shows the signal reconstructed after 30 iterations from a data (Fig.2a) consisting of 10% of the samples of the original signal (Fig.1a). Fig.5b shows the spectrum of the reconstructed signal. Fig.6, shows the variation of the total mean squared error with iterations, with and without the initial interpolation procedure. For a given number of iterations, the error is significantly smaller with initial interpolation than without. Fig.7 shows similar curves for a high fréquency signal (i.e., f_1=140, f_2=150). Note that the intial interpolation, through demodulation, relaxation procedure and remodulation, produces significantly better results compared to the standard method of assuming zero values for the missing samples.

So far we have considered the case of only 10% of samples in the data. The performance for different percentages of samples in the data is illustrated in Fig.8. It shows the plot of the ratio of the mean squared error with and without initial interpolation versus the percentage of samples in the data. The curve shows that the performance is best when the percentage of known samples is within a certain range. This range is different for different sample distributions, but they are all typically in the range of 8% to 15% of samples. These curves bringout the following points:

When the percentage of known samples is too low (5%) or too high (50%), the effect of initial interpolation is not significant. This is because, in the former case, the known samples are too few for any interpolation scheme to work properly. In the latter case the known samples are sufficient to indicate the spectral peak clearly and hence no further spectral peak enhancement would be necessary.

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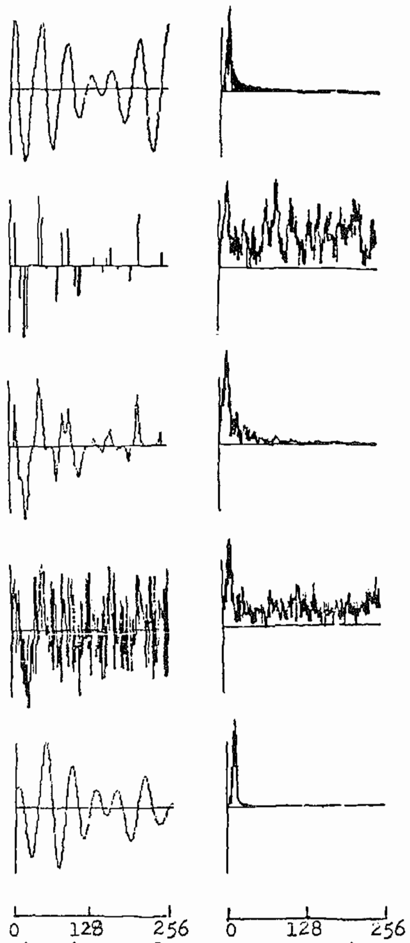

(a) signal (b) magnitude spectrum

Fig.1.The sample signal and its spectrum used in our illustrations.

Fig.2 Partial data derived from the complete signal in Fig.1a and the spectrum of the partial data.

Fig.3.Interpolated data obtained using relaxation procedure on the partial data in Fig.2a and the spectrum of the interpolated data.

Fig.4.Initial interpolated signal obtained by adding the noise component to the relaxation interpolated data,and the spectrum of the signal.

Fig.5.Signal reconstructed from the initial interpolated data in Fig.4a using Papoulis-Gerchberg algorithm and the spectrum of the reconstructed signal

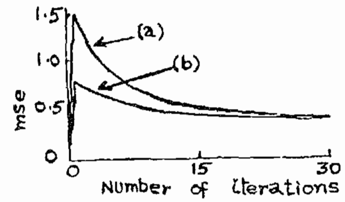

Fig.6. For the low frequency signal case of partial data(Fig.2a), the mean squared error in the reconstructed signal as a function of number of iterations.
(a)without initial iterpolation
(b)with initial interpolation

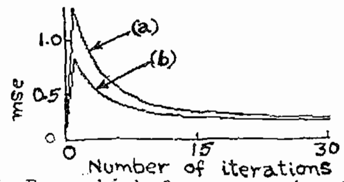

Fig.7. For a high frequency signal case of partial data, the mean squared error in the reconstructed signal as a function of number of iterations
(a)without (b) with, initial interpolation

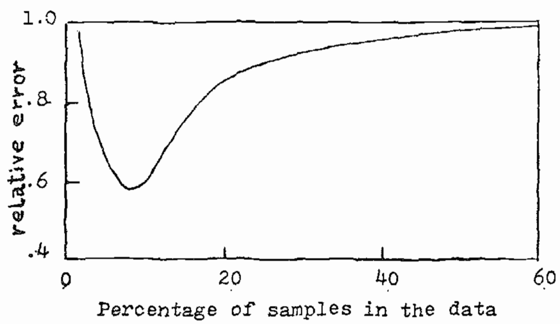

Fig.8. Relative error as a function of percentage of samples in the data. The relative error is computed as the ratio of the mean squared errors in the reconstructed signals with and without initial interpolation.