Waveform Estimation Using Group Delay Processing

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***Abstract*—A method of signal waveform estimation from an ensemble of jittered noisy measurements is presented. The method uses group delay functions to perform the ensemble averaging and thus overcomes the difficulty of computing the unwrapped phase function before averaging. We propose a new technique, called group delay processing, to estimate the signal waveform if only a single noisy measurement is available. We demonstrate our group delay averaging and group delay processing techniques through illustrative examples.**

I. Introduction

THE objective of this paper is to present a method of recovering a signal from an ensemble of noisy measurements in which the signal has a variable delay. This problem was addressed by Rodriguez *et al.* in [1]. They proposed an unwrapped phase averaging technique to solve the problem. In this paper we show that computation of the unwrapped phase can be avoided using group delay functions [2]. We also demonstrate how processing in the group delay domain helps to reduce the effect of additive noise.

The problem can be stated as follows. Given an ensemble of noisy measurements

$$r_i(t) = s(t - D_i) + n_i(t), \qquad i = 1, 2, \cdots, M, \qquad 0 \le t \le T \tag{1}$$

where $s(t)$ is a deterministic signal, D_i is a random delay, and $n_i(t)$ is the additive noise, the objective is to determine the signal $s(t)$ from $\{r_i(t)\}$.

If $D_i = 0$, then a simple ensemble averaging of $\{r_i(t)\}$ can be used as an estimate of the signal. If $D_i \neq 0$, then the signal jitter results in a convolutionally distorted signal. The effect of signal jitter can be eliminated by averaging the signals in the frequency domain. This method was suggested by Rodriguez *et al.* in [1], where ensemble averaging of the Fourier transform magnitude and the unwrapped phase was performed to reduce the effects of distortions introduced by noise and jitter in the measurements.

There are basically two problems in the frequency domain averaging procedure for estimation of signal waveform. The first is the computation of the unwrapped phase function and the second is the effect of noise. The phase of the noise is likely to play a major role in the phase averaging, especially in the frequency regions where the noise spectrum $|N_i(\omega)|$ is greater than the signal spectrum $|S(\omega)|$.

We propose a method which eliminates the need for computing the unwrapped phase. We show that the signal waveform can be estimated as accurately as in [1] by using group delay functions [2]. We then show that we can reduce the effect of noise by processing in the group delay domain.

In Section II we introduce the group delay functions and show how they can be used in the place of unwrapped phase function for ensemble averaging. In Section III we illustrate the results obtained by averaging the group delay from phase and compare our results with those obtained by unwrapped phase averaging, as given in [1]. The quality of the estimated signal waveform by ensemble averaging in the frequency domain is poor if the additive noise is white instead of colored, as discussed in [1]. We discuss the case of additive white noise in Section IV and show that, by processing in the group delay domains, we can reduce the effect of the low signal-to-noise ratio (SNR) regions of the spectrum on the estimated signal waveform.

II. Group Delay Functions

For a discrete time signal $\{s(n)\}$, we define the Fourier transform as

$$S(\omega) = \sum_{n=0}^{\infty} s(n) \exp(-j\omega n) = |S(\omega)| \exp(j\theta(\omega)) \tag{2}$$

where $\theta(\omega)$ is the unwrapped phase function [3]. Let

$$\ln |S(\omega)| = \sum_{n=0}^{\infty} c_1(n) \cos n\omega \tag{3}$$

$$\theta(\omega) = -\sum_{n=0}^{\infty} c_2(n) \sin n\omega. \tag{4}$$

Then we define the group delay from phase as [2]

$$\tau_p(\omega) = \sum_{n=1}^{\infty} nc_2(n) \cos n\omega. \tag{5}$$

Analogously, we define the group delay from magnitude as

$$\tau_m(\omega) = \sum_{n=1}^{\infty} nc_1(n) \cos n\omega. \tag{6}$$

If $\phi_{\min}(\omega)$ is the unique phase function such that the inverse Fourier transform of $|S(\omega)| \exp(j\phi_{\min}(\omega))$ is a min-

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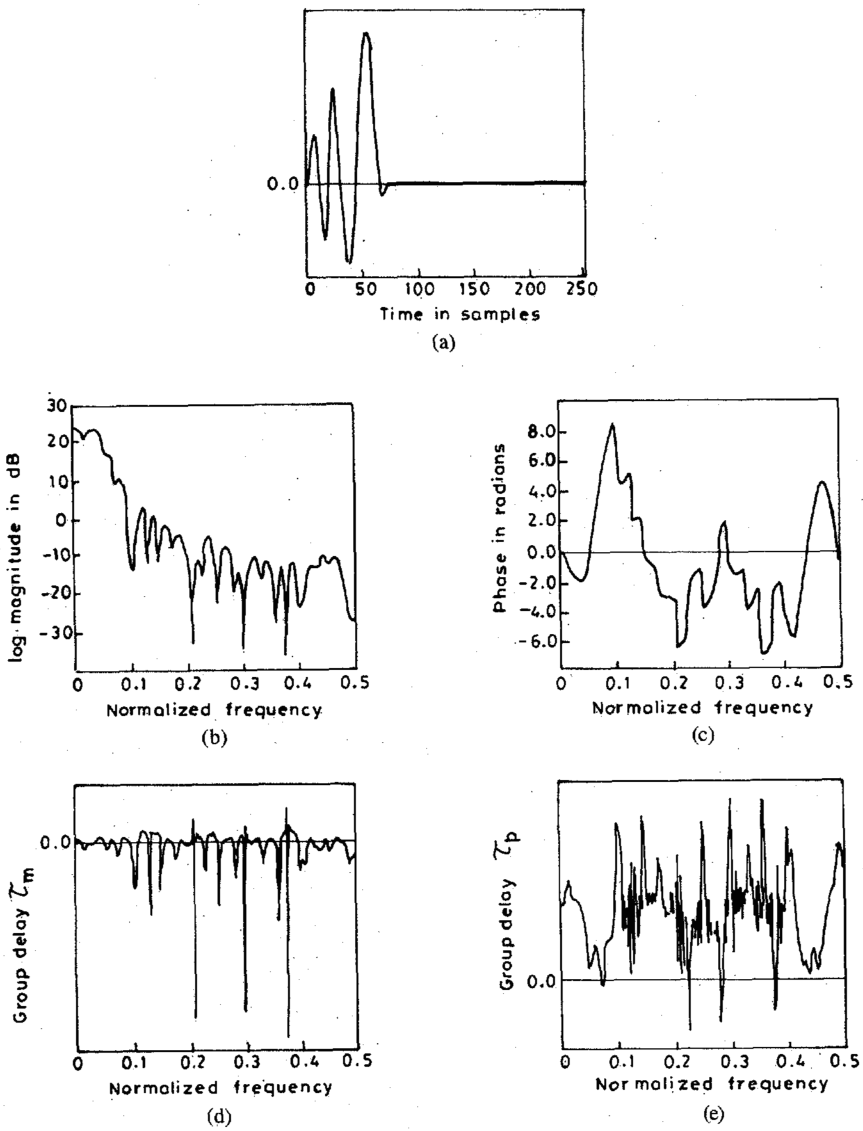

Fig. 1. Signal used in simulations. (a) Time domain signal. (b) Log magnitude. (c) Unwrapped phase with linear phase component removed. (d) Group delay from magnitude ($\tau_m(\omega)$). (e) Group delay froom phase ($\tau_p(\omega)$).

imum phase signal, then $\tau_m(\omega)$ can be shown to be the negative derivative of $\phi_{\min}(\omega)$. $\tau_m(\omega)$ can be computed by finding $\{c_1(n)\}$ from (3). $\tau_p(\omega)$ cannot be computed from (4) and (5) unless we have the unwrapped phase function $\theta(\omega)$. However, $\tau_p(\omega)$ can be directly calculated using the all-pass sequence derived from phase as in [3] and [4]. From the wrapped phase $\phi(\omega)$, $\tau_p(\omega)$ is calculated as follows. Let

$$\exp(j\phi(\omega)) = G_1(\omega) + jG_2(\omega) = G(\omega). \tag{7}$$

Then the inverse Fourier transform of $G(\omega)$ is given by

$$g(n) = F^{-1}[\exp(j\phi(\omega))], \qquad n = 0, 1, \cdots, N-1 \tag{8}$$

where $g(n)$, $n = 0, 1, \cdots, N/2$ correspond to the positive time samples and $g(n)$, $n = N/2 + 1, N/2 + 2, \cdots, N - 1$ correspond to the negative time samples. Form the sequence

$$\begin{aligned} h(n) &= ng(n), \qquad n = 0, 1, \cdots, N/2 \\ &= (n - N)\, g(n), \qquad n = N/2 + 1, \\ &\qquad N/2 + 2, \cdots, N - 1. \end{aligned}$$

Then the Fourier transform of $\{h(n)\}$ is given by

$$F[h(n)] = H_1(\omega) + jH_2(\omega) = H(\omega). \tag{9}$$

The group delay from phase is given by [3]

$$\tau_p(\omega) = G_1(\omega)\, H_1(\omega) + G_2(\omega)\, H_2(\omega). \tag{10}$$

III. Signal Waveform Estimation Using Group Delay Averaging

We use group delay averaging instead of unwrapped phase averaging for estimating the signal waveform and compare our results with those given in [1]. In Fig. 1(a)

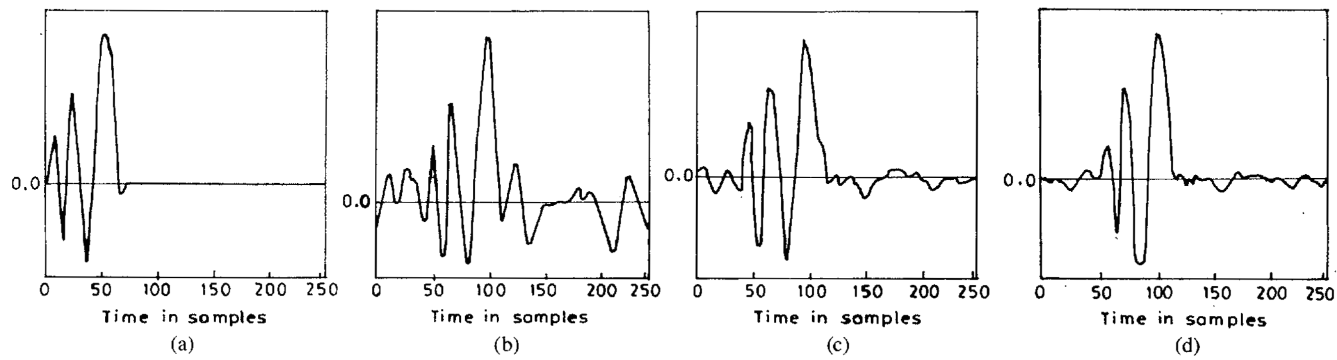

Fig. 2. Comparison of signal waveforms estimated by group delay averaging and by unwrapped phase averaging. (a) Original time domain signal. (b) A measurement sample (noisy signal with random delay). (c) Signal recovered by unwrapped phase averaging. (d) Signal recovered by group delay averaging.

we show 256 samples of the basic signal chosen for simulations. Fig. 1(b) and (c) shows the log magnitude and unwrapped phase functions and Fig. 1(d) and (e) shows the corresponding $\tau_m(\omega)$ and $\tau_p(\omega)$, respectively. For the purpose of simulation, the signal was given a random delay D with a uniform distribution of 15 units wide and with an average delay of 48. Colored noise having an average spectral density $\overline{\beta}$ times that of the signal was added to produce the measurement ensemble $r_i(n)$, $i = 1, 2, \cdots, M$. For each measurement $r_i(n)$, $\tau_{pi}(\omega)$ is calculated form the (wrapped) phase $\phi_i(\omega)$. When zeros of the z-transform of $\{r_i(n)\}$ occur on the unit circle in the z-plane, $\tau_{pi}(k)$, which represents $\tau_{pi}(\omega)$ at N discrete frequencies, is poorly sampled, but can nevertheless be computed. The phase unwrapping algorithm proposed in [6], however, cannot accept zeros on the unit circle in any of the measurements $r_i(n)$, $i = 1, 2, \cdots, M$.

The ensemble averaged group delay $\overline{\tau}_p(\omega)$ is calculated as

$$\overline{\tau}_p(\omega) = \frac{1}{M}\sum_{i=1}^{M} \tau_{pi}(\omega). \tag{11}$$

The average phase $\overline{\theta}(\omega)$ is computed through $\{c(n)\}$ using the relations (11), (5), and (4). The empirical bias correction proposed in [1] is not used here. The estimated unwrapped phase $\hat{\theta}(\omega)$ is given by

$$\hat{\theta}(\omega) = \overline{\theta}(\omega). \tag{12}$$

Let

$$\beta(\omega) = \frac{\overline{|N_i(\omega)|^2}}{|S(\omega)|^2} \tag{13}$$

where the bar denotes ensemble average. For a case where the signal and noise have similarly shaped spectral densities, we get

$$\overline{\beta} = 1/\mathrm{SNR} = \frac{\dfrac{1}{M}\displaystyle\sum_{i=1}^{M}\sum_{n=0}^{N-1} n_i^2(n)}{\displaystyle\sum_{n=0}^{N-1} s^2(n)}. \tag{14}$$

The estimate of the magnitude spectrum $|\hat{S}(\omega)|^2$ is given as in [1] by

$$|\hat{S}(\omega)|^2 = \frac{1}{1+\overline{\beta}}\overline{|R_i(\omega)|^2}. \tag{15}$$

Using (12) and (15) we get the estimated signal waveform $\hat{s}(n)$ as

$$\hat{s}(n) = F^{-1}[|\hat{S}(\omega)| \exp(j\hat{\theta}(\omega))]. \tag{16}$$

The estimated signal waveforms obtained by our group delay averaging method and by the unwrapped phase averaging method are compared in Fig. 2(a) to (d) for $M = 10$ and SNR = 5 dB. Fig. 3 shows another example of waveform estimation by the group delay averaging and by the unwrapped phase averaging methods. These examples demonstrate that computation of the unwrapped phase function as suggested in [1] can be avoided. The group delay averaging method gives comparable results.

IV. Noise Suppression by Group Delay Processing

In order to be able to write (15) for $|\hat{S}(\omega)|$, it was assumed in [1] that the signal and the noise have similarly shaped spectral densities so that $\beta(\omega)$ in (13) could be taken as a constant equal to $\overline{\beta}$. This is an artificial restriction and is not justified in practice. The result of waveform estimation from signals corrupted by white noise is shown in Fig. 4. The figure shows that the waveform estimate is poor for a white noise situation. We propose a group delay processing method to reduce the effect of noise on the estimated waveform. The method uses a smoothed $\tau_m(\omega)$ to identify the high SNR regions of the spectrum. The signal waveform is estimated using the group delay functions in the high SNR regions. The information contained in the low SNR regions of the spectrum is removed from the group delay functions before reconstruction. Equation (6) for $\tau_m(\omega)$ can be rewritten as

$$\tau_m(\omega) = \sum_{n=1}^{N/2-1} nc_1(n)\cos n\omega. \tag{17}$$

A smoothed $\tau_m(\omega)$ can be obtained by taking only the first

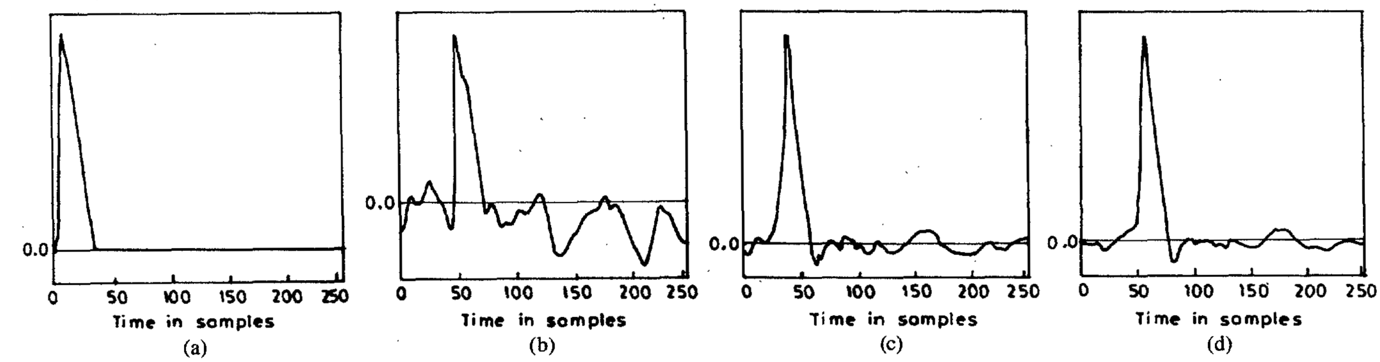

Fig. 3. Illustration of waveform estimation by group delay averaging for another signal. (a) Original time domain signal. (b) A mesurement sample (noisy signal with random delay). (c) Signal recovered by unwrapped phase averaging. (d) Signal recovered by group delay averaging.

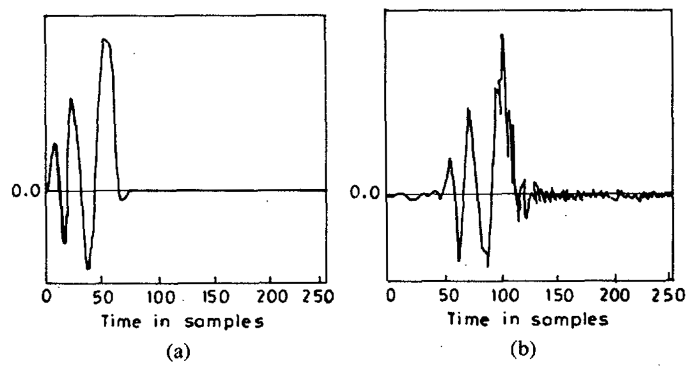

Fig. 4. (a) Original time domain signal. (b) Ensemble averaged signal for the case of white noise. Number of measurement samples $M = 10$. SNR = 10 dB.

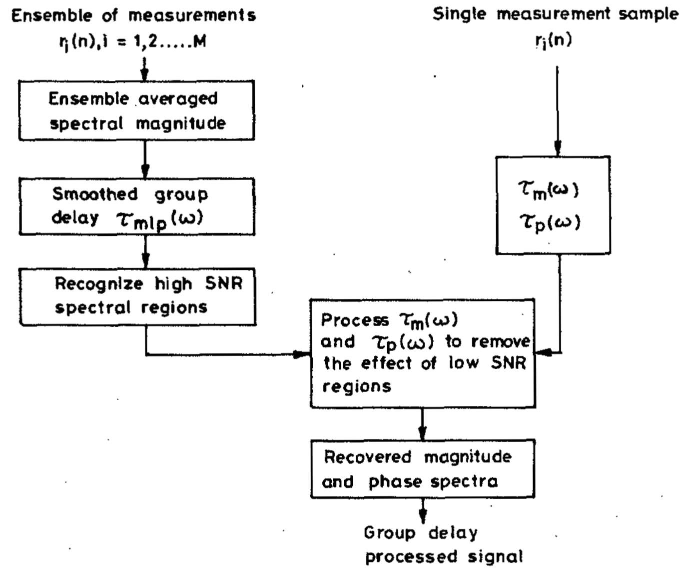

Fig. 5. Block diagram illustrating steps involved in group delay processing.

few cepstral coefficients [5]. Denoting the smoothed $\tau_m(\omega)$ by $\tau_{mlp}(\omega)$, we have

$$\tau_{mlp}(\omega) = \sum_{n=1}^{M1} nc_1(n) \cos n\omega. \tag{18}$$

In order to avoid spurious peaks due to truncation, it is necessary to appropriately window the cepstrum in (18). We have used a linearly tapering window for the end samples. The positive and negative regions of $\tau_{mlp}(\omega)$ correspond approximately to the high and low SNR regions of the spectrum, respectively. The effect of the low SNR regions upon the time domain signal must be minimized. This is achieved by setting these regions in $\tau_m(\omega)$ and $\tau_p(\omega)$ to the average values within those regions, taking care to see that there are no abrupt discontinuities in the resulting group delay functions at the edges of the selected frequency regions. The group delay processing technique is summarized in Fig. 5.

It is possible to process even the individual measurements in an ensemble using this technique. The results of such processing are shown in Fig. 6. The smoothed $\tau_m(\omega)$ and the processed signal for $M1 = 5$ for one sample function of the measurement ensemble are shown in the figure. The choice of $M1$ is in general dictated by the shape of the spectral envelope. The results show that the group delay processing helps in reducing the effect of noise. If an ensemble consisting of a large number of measurements is available, then the high SNR regions can be estimated from the smoothed $\tau_m(\omega)$ derived from the ensemble averged magnitude spectrum. Fig. 7 shows the result of processing the measurement sample of Fig. 6(a) with the knowledge of the high SNR regions derived from the ensemble averaged signal in Fig. 4(b). The figure shows that estimating the high SNR regions from the ensemble of measurements yields better results than from a single measurement. These results indicate that group delay processing yields a better estimate of the signal waveform compared to the estimate from ensemble averaging for additive white noise situation.

V. Conclusions

We have shown that computation of the unwrapped phase for ensemble averaging of noisy measurement can be eliminated by using the group delay functions. The results obtained by the group delay averaging are similar to those obtained by the unwrapped phase averaging [1]. We have found that the estimated waveform is generally poor if the additive noise is white. We have also noticed that it

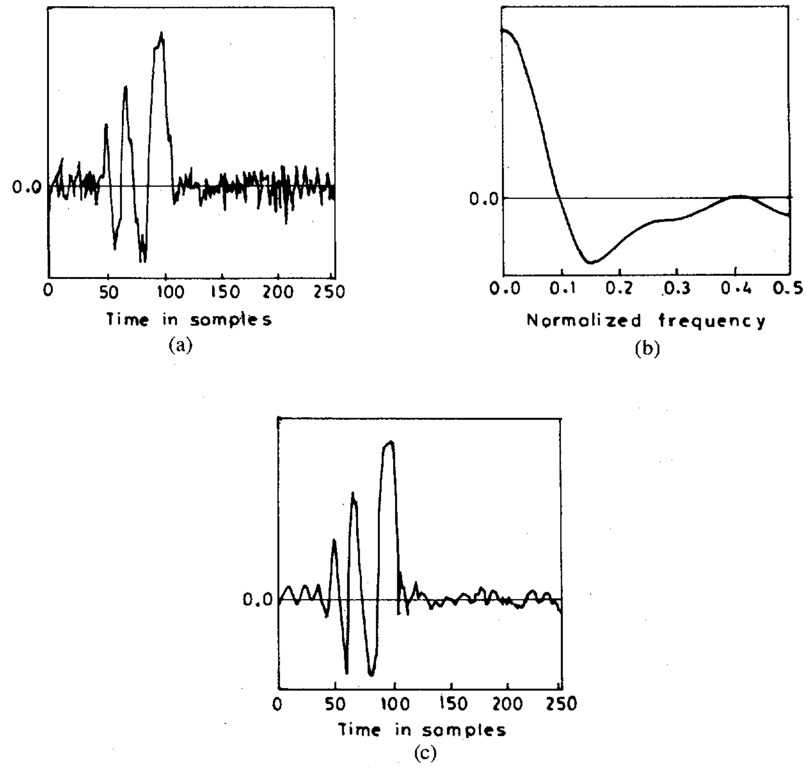

Fig. 6. Group delay processing of a single mesurement for SNR = 10 dB. (a) A mesurement sample. (b) Smoothed $\tau_m(\omega)$ for $M1 = 5$. (c) Group delay processed signal.

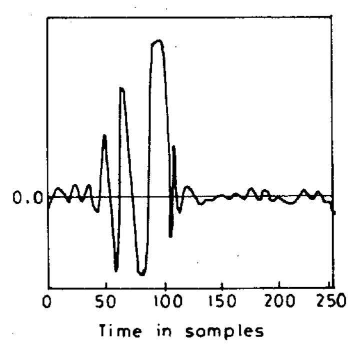

Fig. 7. Group delay processing of the signal in Fig. 6(a) using an estimate of high SNR spectral regions derived from the ensemble averaged ($M = 10$, SNR = 10 dB) signal in Fig. 4(b).

is difficult to perform phase unwrapping reliably for noisy signals. We have demonstrated that the effect of noise can be reduced by processing the noisy signals in the group delay domain. The estimated waveform can be significantly improved by recognizing the high SNR regions from the ensemble averaged signal.

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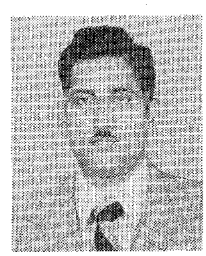

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J. Sreekanth, biography and photograph not available at time of publication.

Anand Rangarajan, biography and photograph not available at time of publication.