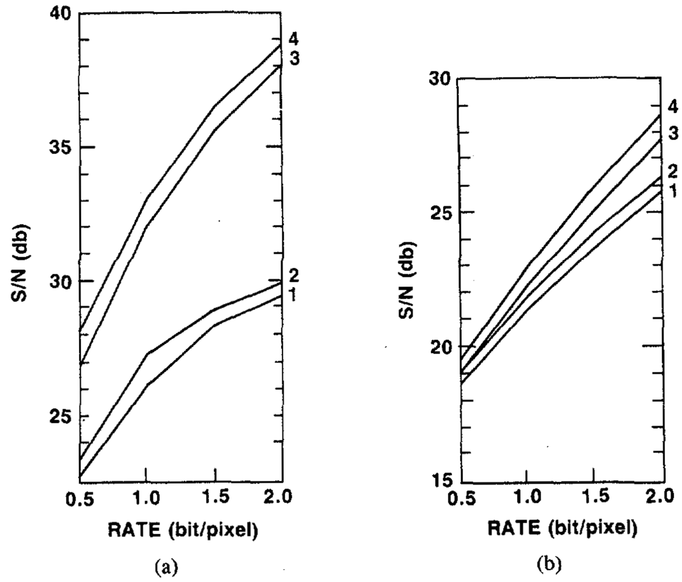

Fig. 1. Simulation results for (a) "Cronkite" and (b) "Plant." Curves 1 and 2: nonuniform Gaussian quantizer. Curves 3 and 4: pdf-optimized quantizer. Curves 1 and 3: rate-distortion theoretic approach. Curves 2 and 4: computational approach.

allocation in this case. As far as the computational efficiency is concerned, the proposed algorithm requires less computations. For $N = 64$, for an average 2 bits per element allocated and for up to 24 bits maximally allowed for any element, Shoham and Gersho's algorithm requires 212 operations (addition, multiplication, and computation) per element. On the other hand, the proposed algorithm requires about 70 operations per element including overhead computations for presorting the variances.

The fast design algorithm described here has been shown to be much more efficient than any previous algorithm. The factor of saving in computation is about $N/7$, which is rather significant in most transform image coding. The effect of non-Shannon quantization error is also incorporated in the computational approach and mild improvement in performance is achieved. The computational complexity of the rate-distortion approach is independent of the number of total available bits. On the other hand, the complexity of the computational approach increases linearly with the number of total available bits. In low bit-rate environment, the fast computational approach may be more efficient than the rate-distortion theoretic approach.

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Representation of Images Through Group-Delay Functions

B. YEGNANARAYANA AND ARVIND RAGHUNATHAN

Abstract—**In this correspondence we propose a new representation for images through the use of group-delay functions. We provide algorithms for the computation of the group-delay functions and for the recovery of images from them. We give several examples to show that most of the perceptually significant information of an image is retained in this representation. We derive the minimum phase equivalent images from the Fourier transform (FT) magnitude, as well as from the phase through their respective group-delay functions. We compare these images to those obtained through iterative reconstruction from the FT magnitude and phase, respectively.**

I. INTRODUCTION

The most common way of representing an image is in the spatial domain as a two-dimensional array of positive numbers, representing gray levels of pixels. An image is also represented in the frequency domain as the Fourier transform of the pixels [1]. The Fourier representation involves complex numbers and, hence, a magnitude part and a phase part. Most processing methods involve manipulating the data in one of the representations. For example, in spectral estimation [2], we model the magnitude spectrum using autoregressive (AR), or autoregressive moving average (ARMA), or moving average (MA) models. One common feature of many of the earlier processing methods in the frequency domain is that they tend to ignore the phase. It has recently been shown [3] that a multidimensional signal can be uniquely specified up to a scale factor by the phase of the Fourier transform alone under centain restrictions.

These studies illustrate the relative importance of the FT magnitude and phase of a signal under different situations. But it is difficult to visualize how the significant information is embedded in the FT magnitude and phase. More importantly, it is difficult to visualize how the information in these two components are related, because the magnitude and the phase are not comparable quantities.

The use of group-delay functions to represent signals offers a solution to the problem. Conventionally, for one-dimensional signals, group-delay is defined as the negative derivative of the unwrapped phase function $\phi(\omega)$ [4]. However, one can define two group-delay functions for a signal—one derived from the spectral magnitude and the other from the phase—as shown in [5]. We denote these as τ_m and τ_p, respectively. The properties of these functions are discussed in [5]. Here we state some properties of these

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functions which offer insight into the problem of image reconstruction.

1) For a minimum phase signal, $\tau_m(\omega) = \tau_p(\omega)$.

2) For a maximum phase signal, $\tau_m(\omega) = -\tau_p(\omega)$.

3) For a mixed phase signal, $|\tau_m(\omega)| \neq |\tau_p(\omega)|$.

A mixed phase signal can be thought of as a convolution (in the time domain) of its minimum and maximum phase components. This corresponds to addition in the group-delay domain.

We develop the group-delay functions applicable for two-dimensional signals in Section II, and discuss algorithms for their computation and for the recovery of images from them. We give several examples of images (each 32 × 32 pixels) recovered from their group-delay functions to demonstrate the validity of this representation. In Section III we derive the minimum phase equivalent images from spectral magnitude and from phase through their respective group-delay functions. We compare these images to those obtained through iterative reconstruction from the Fourier transform magnitude and phase, respectively.

II. Theory of Group-Delay Functions

An image is a two-dimensional signal represented as a two-dimensional sequence or numbers $x(n_1, n_2)$, $n_1 = 0, 1, 2, \cdots, N_1 - 1$, and $n_2 = 0, 1, 2, \cdots, N_2 - 1$. The Fourier transform of $\{x(n_1, n_2)\}$ is given by

$$X(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2)\, e^{-j\omega_1 n_1}\, e^{-j\omega_2 n_2}$$
$$= |X(\omega_1, \omega_2)|\, e^{j[\theta(\omega_1,\omega_2) + 2\pi\lambda(\omega_1,\omega_2)]} \tag{1}$$

where $\theta(\omega_1, \omega_2)$ is the principal value of phase and $\lambda(\omega_1, \omega_2)$ is an integer such that the overall phase becomes a continuous function of ω_1 and ω_2. The function

$$\phi(\omega_1, \omega_2) = \theta(\omega_1, \omega_2) + 2\pi\lambda(\omega_1, \omega_2) \tag{2}$$

is called the unwrapped phase function. It is assumed that $|X(\omega_1, \omega_2)| \neq 0$ for all ω_1 and ω_2. Consequences of this assumption will be discussed later.

Since there are two freqeuncy variables ω_1 and ω_2, we define two group-delay functions from phase as follows:

$$\tau_{p1} = -\frac{\partial\phi(\omega_1, \omega_2)}{\partial\omega_1} \quad \text{and} \quad \tau_{p2} = -\frac{\partial\phi(\omega_1, \omega_2)}{\partial\omega_2}. \tag{3}$$

Although minimum phase function in two dimensions is not well understood, we use the term analogous to the one-dimensional case. In particular, we call the phase function derived from the FT magnitude as the minimum phase function, and the signal obtained from the FT magnitude and the minimum phase function as the minimum phase equivalent signal from spectral magnitude. Let $\phi_{\min}(\omega_1, \omega_2)$ be the unique minimum phase function corresponding to $|X(\omega_1, \omega_2)|$, then we define two group-delay functions from $\phi_{\min}(\omega_1, \omega_2)$ as follows:

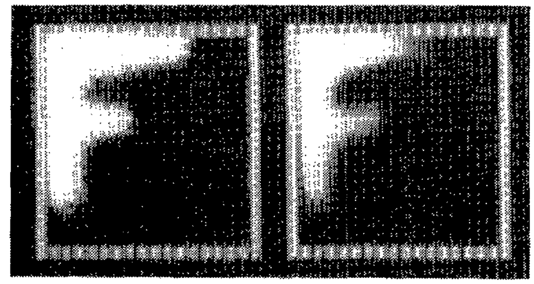
$$\tau_{m1} = -\frac{\partial\phi_{\min}(\omega_1, \omega_2)}{\partial\omega_1} \quad \text{and} \quad \tau_{m2} = -\frac{\partial\phi_{\min}(\omega_1, \omega_2)}{\partial\omega_2}. \tag{4}$$

The term "unique minimum phase function" is used here in the sense that the cepstral coefficients $\{c(n_1, n_2)\}$ derived from $\ln|X(\omega_1, \omega_2)|$ determine the magnitude and phase functions completely. We now describe algorithms to compute the two-dimensional group-delay functions. The algorithms are based on the computation of the cepstral coefficients used in [6] and [7]. The algorithm for the computation of τ_{m1} and τ_{m2} thus consists of the following steps.

1) Compute $X(\omega_1, \omega_2)$, the Fourier transform of $\{x(n_1, n_2)\}$.

2) Take the inverse Fourier transform of $\ln|X(\omega_1, \omega_2)|$ to obtain $\{c(n_1, n_2)\}$, the cepstral coefficients. Note that whenever $|X(\omega_1, \omega_2)| \leq \epsilon$, then $|X(\omega_1, \omega_2)|$ is set equal to ϵ, where ϵ is a small positive quantity.

3) Form the two-dimensional sequences $\{n_1 c(n_1, n_2)\}$ and $\{n_2 c(n_1, n_2)\}$ and make them even symmetric about the origin.



(a) (b)

Fig. 1. Image reconstruction from group-delay functions: example 1. (a) Original (32 × 32). (b) Signal reconstructed from group-delay functions.

4) Compute the Fourier transforms of these two sequences to obtain τ_{m1} and τ_{m2}, respectively.

To recover the spectral magnitude from τ_{m1} and τ_{m2}, we combine the two sets of cepstral coefficients [7] derived from τ_{m1} and τ_{m2}, respectively, and then compute the Fourier transform which yields the log magnitude. The steps involved in the computation of the spectral magnitude from τ_{m1} and τ_{m2} are as follows.

1) Compute $\{n_1 c_1(n_1, n_2)\}$ and $\{n_2 c_2(n_1, n_2)\}$, the inverse Fourier transform of τ_{m1} and τ_{m2}, respectively.

2) Form the sequences $\{c_1(n_1, n_2)\}$ and $\{c_2(n_1, n_2)\}$ and make them even symmetric.

3) Compute the cepstral coefficients as follows.

a) The first row of cepstral coefficients consists of those derived from τ_{m2} in steps 1) and 2).

b) The first column of cepstral coefficients consists of those derived from τ_{m1} is steps 1) and 2).

c) The rest of the two-dimensional array is computed as the average of the two sets of cepstral coefficients.

4) Take the inverse Fourier transform of the cepstral coefficients to get the log magnitude. Exponentiate it to obtain the spectral magnitude.

Computation of τ_{p1} and τ_{p2} is similar to the one-dimensional case, the algorithm for which is given in [5]. The algorithm is based on the following relations.

$$\tau_{p1} = \frac{X_R Y_{1R} + X_I Y_{1I}}{|X|^2} \quad \text{and} \quad \tau_{p2} = \frac{X_R Y_{2R} + X_I Y_{2I}}{|X|^2} \tag{5}$$

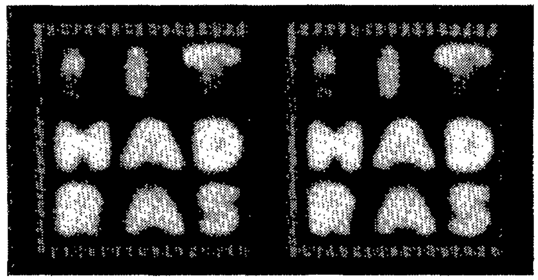
where Y_1 is the Fourier transform on $\{n_1 x(n_1, n_2)\}$ and Y_2 is the Fourier transform of $\{n_2 x(n_1, n_2)\}$. The subscript R stands for the real part of the Fourier transform and the subscript I stands for the imaginary part of the Fourier trasnform. The average values of the group-delays τ_{p1} and τ_{p2} give the linear phase components in the unwrapped phase function $\phi(\omega_1, \omega_2)$ along the axes ω_1 and ω_2, respectively.

Recovery of phase from τ_{p1} and τ_{p2} is similar to the algorithm given above for recovering the log magnitude, with the difference that we make the cepstral coefficients derived from the group-delay functions odd symmetric about the origin. Linear phase terms have to be added along the two axes to recover the phase from the group-delay functions.

Fig. 1(a) shows a picture, and Fig. 1(b) shows the image reconstructed from the group-delay functions derived from the original picture. We notice that the reconstructed image looks almost like the original. Figs. 2 and 3 show the original and the reconstructed images for two other pictures. These examples demonstrate that the group-delay representation is adequate for a wide variety of pictures, despite the problems in computing the group-delay functions for real signals due to very small values of $|X|$ at some frequency points.

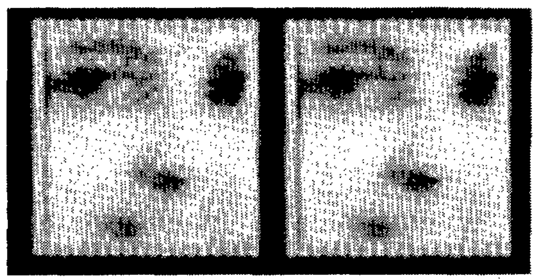
III. Minimum Phase Equivalent Images

We propose in this section noniterative algorithms for the reconstruction of the minimum phase equivalent images from spectral magnitude or phase.



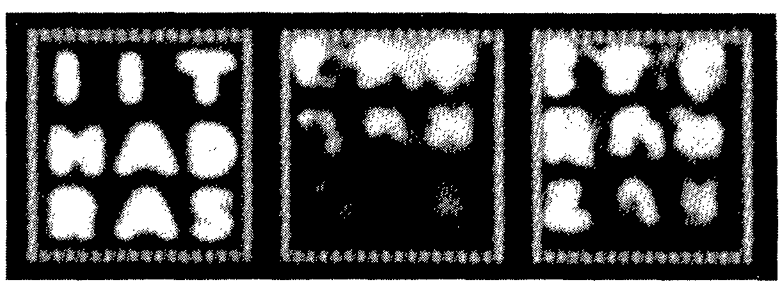
(a) (b)

Fig. 2. Image reconstruction from group-delay functions: example 2. (a) Original (32 × 32). (b) Signal reconstructed from group-delay functions.



(a) (b)

Fig. 3. Image reconstruction from group-delay functions: example 3. (a) Original (32 × 32). (b) Signal reconstructed from group-delay functions.



(a) (b) (c)

Fig. 4. Image reconstruction from FT magnitude. (a) Original. (b) Minimum phase equivalent image from FT magnitude. (b) Iterative image reconstruction from FT magnitude.

The proposed algorithm is as follows.

1) Compute the group-delay functions from the spectral magnitude or phase as the case may be.

2) Set the group-delay functions from phase and magnitude equal. That is, $\tau_{m1} = \tau_{p1}$ and $\tau_{m2} = \tau_{p2}$.

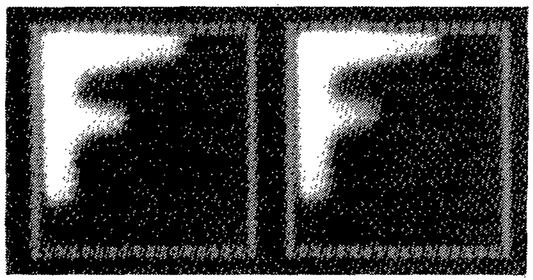
3) Recover the phase and spectral magnitude by the algorithm described in the previous section.

4) Perform an inverse Fourier transform operation to reconstruct the image.

In Fig. 4(a) we show a picture, and in Fig. 4(b) the minimum phase equivalent image reconstructed from the spectral magnitude along for this picture. A confusing picture is obtained, demonstrating that the minimum phase equivalent signal derived from the spectral magnitude alone does not contain sufficient information of the picture. Similar conclusion can be drawn from the image reconstructed iteratively from the spectral magnitude as shown in Fig. 4(c) [3]. In Fig. 5(a) we show another picture which approximates a minimum phase signal, and in Fig. 5(b) we show the image reconstructed from the spectral magnitude alone. It looks almost like the original, showing that, for minimum phase signals (i.e., when most of the signal energy is concentrated near the origin), the spectral magnitude contains most of the significant information. Fig. 6 shows the original picture, the minimum phase equivalent image reconstructed from phase alone, and the image reconstructed iteratively from phase alone by the algorithm described in [3]. Note that the latter two are not identical as discussed in [5] for the one-dimensional case. Fig. 6(b) and Fig. 4(b) also show that for some pictures important features of the original picture are retained in the minimum phase equivalent image derived from phase, but not in the image derived from the spectral magnitude.

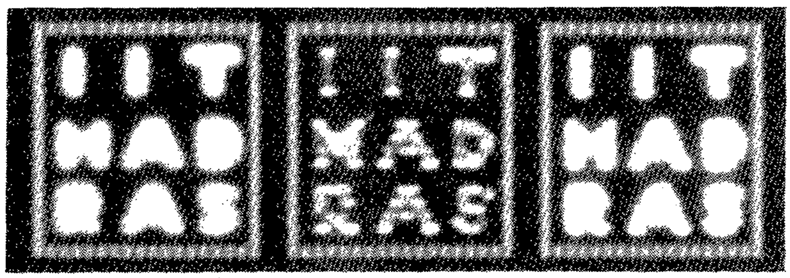
IV. Conclusions

In this correspondence we have proposed a new representation for images through group-delay functions. This representation does not involve loss of significant information. Our results also show the importance of phase in images. There are several computational



(a) (b)

Fig. 5. Image reconstruction from FT magnitude: approximate minimum phase signal. (a) Original. (b) Minimum phase equivalent image from FT magnitude.



(a) (b) (c)

Fig. 6. Image reconstruction from FT phase. (a) Original. (b) Minimum phase equivalent image from FT phase. (c) Iterative image reconstruction from FT phase.

issues which have not been specifically addressed here. In particular, we do not have a good method of handling the cases where the signal spectrum is nearly zero at some frequencies.

The advantage of the group-delay representation is that it allows manipulation of the information contained in phase. We are currently exploring the possibility of using this method of phase processing for image reconstruction, enhancement, and restoration.

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