Design of Recursive Group-Delay Filters by Autoregressive Modeling

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Abstract—A new method of design of recursive group-delay filters is presented in this paper. The method uses autoregressive modeling to derive parameters of the filter. The design is based on the following line of argument. The desired group-delay characteristics are to be realized through a stable all-pass filter. The transfer function of an all-pass filter is completely determined by the coefficients of either the numerator or the denominator polynomial. For a stable all-pass filter, the denominator polynomial must be a minimum phase polynomial. For a minimum phase polynomial, the magnitude function and the group-delay function are related through cepstral coefficients. Therefore, from the given group-delay specification, the cepstral coefficients corresponding to the pole part (inverse of the denominator polynomial) of the desired filter are first determined. The magnitude spectrum corresponding to these cepstral coefficients is approximated using autoregressive modeling. A wide variety of group-delay filters can be realized by the proposed method, depending on the nature of approximation used and the accuracy desired. The design procedure is illustrated through examples.

I. Introduction

GROUP-delay filters are used to realize a given phase response in linear systems. This paper deals with a new method for design of group-delay filters. The key idea upon which the design is based is the relation between cepstral coefficients and the group-delay response of a minimum phase all-pole filter [1]-[4]. The procedure involves first obtaining the cepstral coefficients corresponding to the all-pole part of the given group-delay response. From the cepstral coefficients, the magnitude response of the all-pole filter can be obtained. The magnitude response is then approximated by using an autoregressive model of appropriate order. The autoregressive coefficients determine the parameters of both numerator and denominator polynomials of the group-delay filter transfer function.

Several design procedures exist for realizing a given group-delay response [5], [6]. Most of these procedures use an iterative computer algorithm to obtain the parameters of the filter. These procedures involve solving a set of nonlinear equations, and hence the complexity of the filter design increases with the order of the filter. Simplified iterative design procedures as given in [6] are too restricted. For example, the design in [6] uses only second-order sections to approximate the shape of the given group-delay response. This results in large error at frequencies where there is an abrupt change in the slope of the group-delay response. Moreover, the design does not permit the use of poles of multiple order.

The design method presented in this paper is noniterative in the sense that the parameters of a filter of specified order can be obtained directly by solving a set of autocorrelation normal equations [7]. In Section II, characteristics of group-delay filters are briefly reviewed. The principle of the proposed filter design is presented in Section III. The actual steps involved in the design are given in Section IV and the details of implementation of the design are discussed in Section V. Some examples of the design are discussed in Section VI to illustrate the variety of group-delay filters that can be designed using this new method.

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II. Characteristics of Group-Delay Filters

A group-delay filter is a recursive all-pass filter whose transfer function is given by

$$H(z) = N(z)/D(z) \tag{1}$$

where

$$N(z) = \sum_{k=0}^{N} A_{N-k} z^{-k} \tag{2}$$

and

$$D(z) = \sum_{k=0}^{N} A_k z^{-k} \tag{3}$$

with $A_0 = 1$.

In the above equations, N is the order of the all-pass filter and $z = e^{j\omega}$ where $\omega = 2\pi f$ and f is the frequency (range 0-1) normalized to the sampling frequency f_s. It can easily be shown that $H(z)$ represents an all-pass filter by verifying that

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H(e^{-j\omega}) = 1. \tag{4}$$

For the filter $H(z)$ to be stable, all the poles, i.e., the roots of the denominator polynomial $D(z)$, must lie within the unit circle in the z plane. In other words, $D(z)$ should be a minimum phase polynomial.

From (1), (2), and (3), we have

$$H(e^{j\omega}) = e^{-jN\omega} \frac{\sum_{k=0}^{N} A_k e^{jk\omega}}{\sum_{k=0}^{N} A_k e^{-jk\omega}}. \tag{5}$$

The summation in the numerator is the complex conjugate of the summation in the denominator. Therefore, if $\theta_2(\omega)$ denotes the phase response of the denominator polynomial $D(z)$, then the phase response $\theta_1(\omega)$ of the numerator polynominal $N(z)$ can be simply obtained as

$$\theta_1(\omega) = -N\omega - \theta_2(\omega). \tag{6}$$

The phase response $\theta(\omega)$ of the overall filter $H(z)$ is given by

$$\begin{aligned}\theta(\omega) &= \theta_1(\omega) - \theta_2(\omega)\\ &= -N\omega - 2\theta_2(\omega).\end{aligned} \tag{7}$$

The normalized group delay of the overall filter is given by

$$\begin{aligned}\tau(\omega) &= \frac{-d\theta(\omega)}{d\omega}\\ &= N + 2\theta_2'(\omega)\end{aligned} \tag{8}$$

where

$$\theta_2'(\omega) = \frac{d\theta_2(\omega)}{d\omega} = -\tau_2(\omega) \tag{9}$$

is the negative group delay of the denominator polynomial. Therefore, the group-delay response of the denominator polynomial of the filter can be obtained from the group-delay response of the all-pass filter using the relation

$$\tau_2(\omega) = \frac{-\tau(\omega)}{2} + \frac{N}{2}. \tag{10}$$

It should be noted that the unnormalized group delay as a function of the actual frequency can be obtained by dividing the normalized group-delay function with the sampling frequency f_s. The group-delay function $\tau(\omega)$ has the following property:

$$\tau(\omega) = \tau(-\omega) = \tau(\omega + 2\pi). \tag{11}$$

III. Principle of the Proposed Design

A. Filter Design Problem

Given the desired group-delay function $\tau_d(\omega)$, the objective in the filter design is to determine the parameters of the digital filter $H(z)$ in (1) such that the resulting group-delay response

$$\tau(\omega) \approx \tau_d(\omega). \tag{12}$$

The parameters to be determined are N and A_k, $k = 1, 2, \cdots, N$.

B. Principle

The principle of the proposed design is as follows.

From the given group-delay function, determine the desired group-delay response $\tau_{2d}(\omega)$ of the denominator polynomial of the filter. This can be obtained using the relation (10). That is,

$$\tau_{2d}(\omega) = \frac{\tau_d(\omega)}{2} + \frac{N}{2}. \tag{13}$$

The value of N, which is not known at the design stage, is not necessary for determining the parameters of the filter. This is because the parameters of the denominator polynomial and hence of the filter can be completely determined from the shape of the group-delay function of the denominator polynomial. Any constant group-delay can easily be incorporated by using the appropriate number of delay units.

Let $\bar{\tau}_d(\omega)$ and $\bar{\tau}_{2d}(\omega)$ be the average values of $\tau_d(\omega)$ and $\tau_{2d}(\omega)$, respectively. Then

$$\bar{\tau}_{2d}(\omega) = \frac{-\bar{\tau}_d(\omega)}{2} + \frac{N}{2} \tag{14}$$

and

$$\tau_{2d}(\omega) - \bar{\tau}_{2d}(\omega) = \frac{-\tau_d(\omega) - \bar{\tau}_d(\omega)}{2}. \tag{15}$$

It is convenient to deal with $-(\tau_{2d}(\omega) - \bar{\tau}_{2d}(\omega))$ for deriving the coefficients of the denominator polynomial.

Hence, we define

$$\begin{aligned}\tau_0(\omega) &= -(\tau_{2d}(\omega) - \bar{\tau}_{2d}(\omega))\\ &= \frac{\tau_d(\omega) - \bar{\tau}_d(\omega)}{2}.\end{aligned} \tag{16}$$

It should be noted that $\tau_0(\omega)$ corresponds to the desired group-delay response of the pole part of the all-pass filter. For the all-pass filter to be stable, $\tau_0(\omega)$ should correspond to the group-delay response of a minimum phase filter. Then all the coefficients of the filter can be determined from $\tau_0(\omega)$ using the properties of minimum phase filters.

For a minimum phase filter, the magnitude and phase responses are related through Hilbert transform [8]. We will show presently that the group-delay response and the magnitude response of a minimum phase filter are related through cepstral coefficients. Once the magnitude response of a minimum phase filter is obtained, it can be approximated by autoregressive modeling to determine the coefficients of the filter [7]. The resulting all-pole filter is guaranteed to be stable because of the use of autocorrelation normal equations for solving for the autoregressive coefficients [7].

C. Relation Among Group Delay, Cepstral Coefficients, and Magnitude Spectrum

Let $V(\omega)$ represent the frequency response of a minimum phase all-pole digital filter. $V(\omega)$ will be periodic in ω with period 2π. Since all the poles of a minimum phase filter lie within the unit circle in the z plane, and since the cepstrum of a minimum phase sequence is a causal sequence [8], $\ln V(\omega)$ can be expressed in Fourier series expansion as in (17):

$$\ln V(\omega) = c(0)/2 + \sum_{k=1}^{\infty} c(k)\, e^{-jk\omega} \tag{17}$$

where $\{c(k)\}$ are called cepstral coefficients. Writing

$$V(\omega) = |V(\omega)|\, e^{-j\theta_V(\omega)}, \tag{18}$$

we get the real and imaginary parts of $\ln V(\omega)$ as

$$\ln |V(\omega)| = c(0)/2 + \sum_{k=1}^{\infty} c(k) \cos k\omega \quad \text{(real part)} \tag{19}$$

and

$$\theta_V(\omega) + 2\lambda\pi = \sum_{k=1}^{\infty} c(k) \sin k\omega \quad \text{(imaginary part)} \tag{20}$$

where λ is an integer. Taking the derivative of $\theta_V(\omega)$, we get

$$\theta'_V(\omega) = \sum_{k=1}^{\infty} kc(k) \cos k\omega. \tag{21}$$

$\theta'_V(\omega)$ represents the group delay of the minimum phase all-pole filter. The log magnitude and the group delay are thus related through cepstral coefficients.

D. Derivation of the Filter Coefficients

From the group-delay function $\tau_0(\omega)$ given by (16), we can determine the cepstral coefficients corresponding to the pole part of the filter using the relation (21). From the cepstral coefficients, the log magnitude spectrum, and hence the magnitude spectrum $P(\omega)$ can be obtained using the relations

$$\ln P(\omega) = \ln |V(\omega)|^2$$

$$= K + 2 \sum_{k=1}^{\infty} c(k) \cos k\omega \tag{22}$$

and

$$P(\omega) = \exp \left[K + 2 \sum_{k=1}^{\infty} c(k) \cos k\omega\right] \tag{23}$$

where K is an arbitrary constant.

The autocorrelation coefficients corresponding to $P(\omega)$ can be obtained using the relation

$$P(\omega) = R(0) + 2 \sum_{k=1}^{\infty} R(k) \cos k\omega. \tag{24}$$

Let $\{r(k)\}$ represent the normalized autocorrelation coefficients. Then

$$r(k) = R(k)/R(0), \qquad \text{for } k = 0, 1, 2, \cdots. \tag{25}$$

The values of $\{r(k)\}$ are independent of the value of K in (23), and hence any value of K can be chosen in (23).

The coefficients of an Nth-order autoregressive model for $P(\omega)$ can be obtained by solving the N autocorrelation normal equations given in (26):

$$\sum_{k=1}^{N} A_k r(|i - k|) = -r(i), \qquad i = 1, 2, \cdots, N. \tag{26}$$

These equations are discussed in [7]. The normal equations can be solved using Durbin's algorithm [7].

Once A_k, $k = 1, 2, \cdots, N$ are known, all the parameters of the filter $H(z)$ are known. In general, the approximation to the desired group-delay characteristics will be better for larger values of N.

IV. Design Procedure

The proposed design procedure is illustrated through the following design steps. Actual implementation details of the design are discussed in Section V.

1) Given the desired group-delay response $\tau_d(\omega)$, compute $\tau_0(\omega)$ using (16).

2) Obtain the cepstral coefficients $\{c(k)\}$ from $\tau_0(\omega)$ using the relation

$$\tau_0(\omega) = \sum_{k=1}^{\infty} kc(k) \cos k\omega. \tag{27}$$

3) Compute the pole spectrum $P(\omega)$ using the relation (23) where K is set to 0.

4) Obtain the normalized autocorrelation coefficients using the relations (24) and (25).

5) Solve the autocorrelation normal equations (26) to obtain the filter coefficients A_k, $k = 1, 2, \cdots, N$.

6) The filter is given by

$$H(z) = z^{-M} \frac{\sum_{k=0}^{N} A_{N-k} z^{-k}}{\sum_{k=0}^{N} A_k z^{-k}} \tag{28}$$

where $A_0 = 1$ and M is a fixed delay which is determined by the value of $\tau_d(\omega)$, the order of the filter N and the causality constraint on the filter. But, in general, the value of M is not relevant to the design of the group-delay filter.

V. Implementation of the Group-Delay Filter Design

The design procedure described in Section IV is implemented as shown in Fig. 1. The desired group-delay response is specified by giving the group-delay values at sample points in the frequency domain. The coefficients $\{kc(k)\}$ are computed from the given group-delay response $\tau_0(\omega)$ using inverse discrete Fourier transform (IDFT). $\ln P(\omega)$ is computed from $\{c(k)\}$ using discrete Fourier transform (DFT). In other words, the cosine inverse transform required for computing $\{kc(k)\}$ and the cosine transform required for computing $\{c(k)\}$ are replaced by IDFT and DFT operations, respectively, which can be efficiently implemented using the FFT algorithm. The second block in the figure computes $\{c(k)\}$ from $\{kc(k)\}$. It should be noted that the division by two in (16) and the multiplication of cepstral coefficients by two in (23) nullify, and hence these operations can be omitted. There is also no need to compute the deviation of $\tau_d(\omega)$ from its average value as given in (16). This is because the cepstral coefficients $\{c(k)\}$ are computed using the inverse discrete Fourier transform (IDFT), in which the first coefficient, which gives the average value of $\tau_d(\omega)$, is not used in the computation of the normalized autocorrelation coefficients $\{r(k)\}$ from $\{c(k)\}$.

The input and output data for the first IDFT block are real and symmetric. The realized group-delay response of the filter is computed from the coefficients $\{A_k\}$ using the

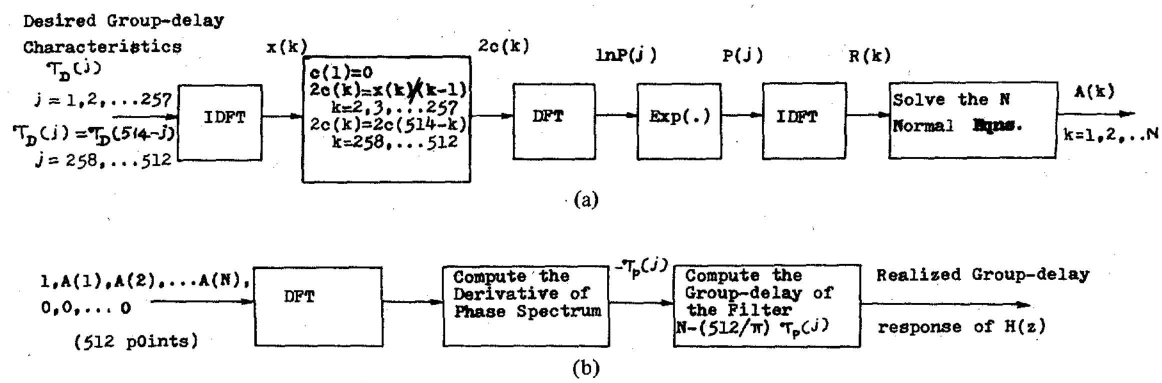

Fig. 1. Block diagram showing the implementation details of the design of group-delay filters by autoregressive modeling. (a) Computation of filter coefficients for a specified order N of the filter $H(z)$. (b) Computation of the group delay response of the realized filter $H(z)$.

algorithm given in [9] for computation of phase spectrum derivatives.

The design examples discussed in the next section were implemented on an IBM 370/155 system (word length, 32 bits).

VI. Design Examples

The procedure described in Fig. 1 is illustrated with some design examples in this section. The normalized frequency range (0-1) is represented by 512 equally spaced points. All DFT and IDFT computations are performed using a 512-point FFT algorithm.

Group-delay characteristics are specified by giving samples of the desired response $\tau_d(\omega)$. Let

$$\tau_D(j) = \tau_d(\omega)\big|_{\omega=2\pi(j-1)/512}$$

and

$$\tau_D(j) = \tau_D(514 - j), \qquad j = 1, 2, \cdots, 257.$$

Then $\tau_D(j)$ forms the input to the design algorithm described in Fig. 1. The true values of the realized group-delay response are obtained by multiplying the derivative of phase spectrum with a scale factor $512/2\pi$ to take into account the effect of sampling in the frequency domain.

Example 1

The first example is the design of a group-delay filter to realize linear group-delay characteristics given by

$$\tau_d(\omega) = 2.5\omega, \qquad 0 \leqslant \omega \leqslant \pi.$$

Then

$$\tau_D(j) = \frac{5\pi(j-1)}{512}, \qquad j = 1, 2, \cdots, 257$$

$$= \tau_D(514 - j), \qquad j = 258, 259, \cdots, 512.$$

Fig. 2 shows the group-delay response of the realized filter for $N=10$ and $N=20$. The desired group-delay characteristics are shown by dotted lines. It is clear that amplitude of ripple is reduced for larger values of N. Also, the error shown in Fig. 3(a) and (b) is nearly uniformly distributed throughout

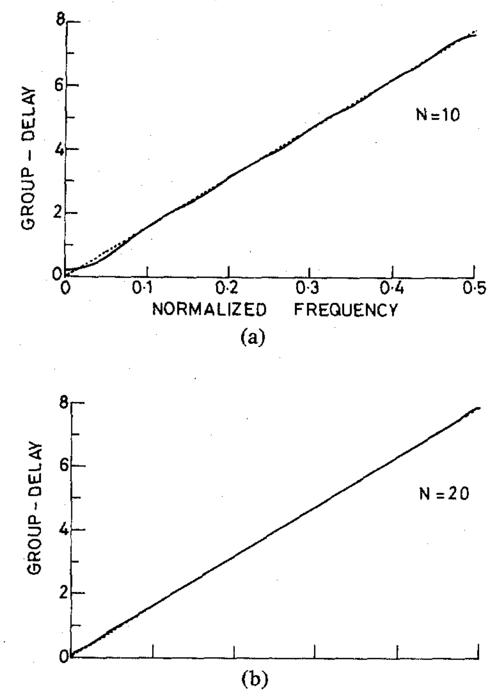

Fig. 2. Realized responses for linear group-delay characteristics. The desired responses are shown by dotted lines. (a) $N = 10$. (b) $N = 20$.

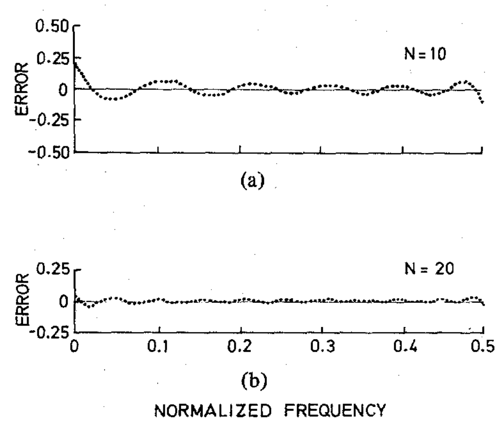

Fig. 3. Difference between realized and desired responses for linear group-delay filters. (a) $N = 10$. (b) $N = 20$.

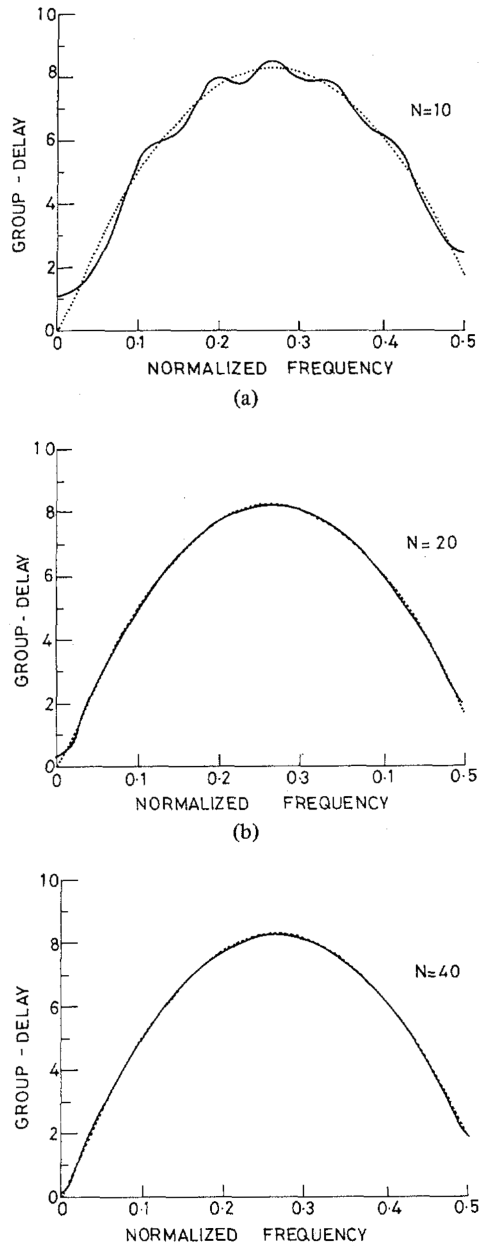

Fig. 4. (a) Realized response for $N = 10$ for quadratic group-delay characteristics. The desired response is shown by the dotted curve. (b) Realized response for $N = 20$ for quadratic group-delay characteristics. The desired response is shown by the dotted curve. (c) Realized response for $N = 40$ for quadratic group-delay characteristics. The desired response is shown by the dotted curve.

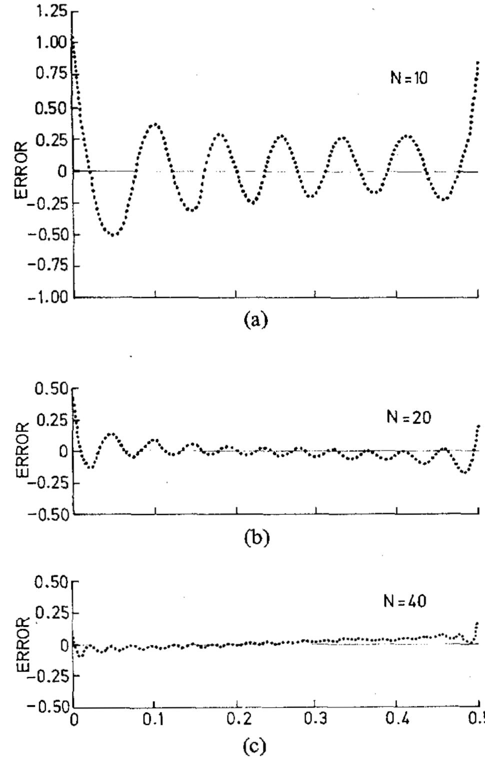

Fig. 5. Difference between realized and desired responses for quadratic group-delay filters. (a) $N = 10$. (b) $N = 20$. (c) $N = 40$.

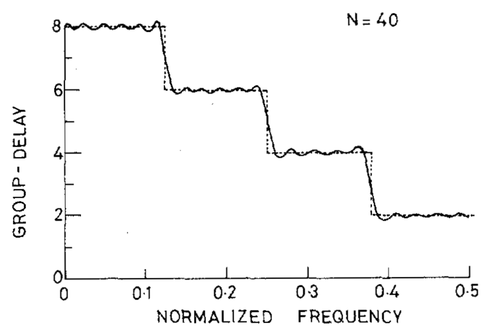

Fig. 6. Realized response for $N = 40$ for stepped group-delay characteristics. The desired response is shown by the dotted curve.

the frequency, unlike in Bernhardt's design [6] where the error is largely near $f = 0$ and $f = 0.5$.

Example 2

The second example is the design of a filter to realize quadratic group-delay characteristics given by

$$\tau_d(\omega) = (10\omega - 3\omega^2), \qquad 0 \leqslant \omega \leqslant \pi.$$

Then

$$\tau_D(j) = \frac{\pi(j-1)}{512}\left(10 - 3 \times \frac{2\pi(j-1)}{512}\right),$$

$$j = 1, 2, \cdots, 257.$$

$$= \tau_D(514 - j), \qquad j = 258, 259, \cdots, 512.$$

Fig. 4 shows the group-delay response of the realized filter for $N = 10$, $N = 20$, and $N = 40$. The desired group-delay characteristics are also shown by dotted curves in each figure. The error curves for this example are shown in Fig. 5(a), (b), and (c). It appears that our design method yields nearly equiripple response whose amplitude can be reduced by increasing the order of the filter.

Example 3

In the next example, the performance of our design method is studied for stepped group-delay characteristics given by

$$\tau_D(j) = 8, \qquad j = 1, 2, \cdots, 65$$

$$= 6, \qquad j = 66, 67, \cdots, 129$$

$$= 4, \qquad j = 130, 131, \cdots, 193$$

$$= 2, \qquad j = 194, 195, \cdots, 257$$

$$= \tau_D(514 - j), \qquad j = 258, 259, \cdots, 512.$$

The realized group-delay response for $N = 40$ is shown in Fig. 6. The amplitude of ripple can be significantly reduced if the transition at each step is made more gradual, i.e., if the transition width is increased.

These examples illustrate that the proposed design method can be used to realize a wide variety of group-delay characteristics.

VII. Conclusions

We have presented a new method of designing recursive group-delay filters. The method can be used to design filters of very high order without additional complexity. This is in contrast to the presently available methods which involve solutions consisting of a set of nonlinear equations that arise as a result of minimizing some error criterion. Usually the solution of the nonlinear equations uses an iterative algorithm which is computationally complex and time-consuming for the design of high order ($N > 10$) filters. Our method yields better results than the design of high-order filters proposed by Bernhardt in [6] because we do not make any simplified assumptions on the group-delay function. Moreover, the restriction of second-order sections in Bernhardt's design is not present in our design. In other words, the realized filter in our design can have real poles and also poles of multiple order. This is the reason for our getting a smaller error, even near $f = 0$ and $f = 0.5$, compared to Bernhardt's results.

It is possible to design a wide variety of group-delay filters, depending on how the desired group-delay function is handled. Since summation of group delays is equivalent to cascading the corresponding filters, the given group-delay characteristics can be split appropriately to obtain a filter in the desired fashion. For example, finite word-length effects of high-order filters can be reduced by splitting the given group-delay into some convenient components and realizing each component independently. This would also place a less severe restriction on the accuracy of the normalized autocorrelation coefficients for high-order filters, which might otherwise result in unstable filters.

Although we have proposed an effective method for realizing a given group-delay characteristic, specific design curves have not been obtained to determine the order of the filter for a given error tolerance. Such design curves for the group-delay filters of this paper and for magnitude response filters of [2] enable us to design filters for given magnitude and phase characteristics.

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