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EFFECTIVENESS OF REPRESENTATION OF SIGNALS THROUGH GROUP DELAY FUNCTIONS

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Abstract. The representation of signals in the group delay domain has been suggested in the literature recently. Because of some special properties of the group delay functions, this representation offers some advantages in several signal processing situations like digital filtering, pole-zero decomposition and deconvolution. In this paper, we study the effectivenesss and limitations of the group delay representation of signal information. We show that most of the limitations arise from the discrete nature of handling the signals in the time and frequency domains. We also show that, as the number of DFT points is increased, the signal derived from the group delay functions, though our reconstruction algorithms, approaches the original signal. We discuss the limitations of the group delay functions in terms of location of the roots of the z-transform of the given discrete time signal. The group delay functions provide an accurate representation of the signal information, as long as the roots are not too close to the unit circle in the z-plane. The errors for the case of close roots are mostly due to reconstruction of phase from the group delay function.

Zusammenfassung. Ein Vorschlag zur Signaldarstellung im Gruppenlaufzeit-Bereich wurde kürzlich veröffentlicht. Wegen einiger besonderer Eigenschaften der Laufzeitfunktionen bietet diese Darstellung bei diversen Signalverarbeitungs-Aufgaben gewisse Vorteile; z.B. bei der digitalen Filterung, der Pol-Nullstellen-Zerlegung und der Entfaltung. Im folgenden Aufsatz untersuchen wir die Wirksamkeit und die Grenzen der Gruppenlaufzeit-Darstellung von Signalinformationen. Wir zeigen, daß die Grenzen im wesentlichen durch die Diskretisierung des Signals im Zeit- und Frequenzbereich hervorgerufen werden. Wir zeigen auch, daß mit Hilfe unserer Algorithmen aus den Gruppenlaufzeit-Funktionen ein Signal rekonstruiert werden kann, das bei Vermehrung der DFT-Werte das Originalsignal immer besser approximiert. Wir diskutieren die Begrenzungen der Gruppenlaufzeit-Funktionen bezüglich der Lage der Nullstellen der z-Transformierten des gegebenen diskreten Zeitsignals. Die Gruppenlaufzeit-Funktionen liefern eine genaue Darstellung der Signalinformation, solange diese Nullstellen nicht zu dicht am Einheitskreis der z-Ebene liegen. Fehler im Falle zu dicht am Kreis liegender Wurzeln sind hauptsächlich auf die Rekonstruktion der Phase aus der Gruppenlaufzeit zurückzuführen.

Résumé. La représentation des signaux dans le domaine du retard de groupe a été récemment suggérée dans la littérature. Grâce à certaines propriétés spéciales des fonctions de retard de groupe, cette représentation offre des avantages dans plusieurs situations de traitement des signaux telles que filtrage numérique, décomposition pôles-zéros et déconvolution. Dans cet article, nous étudions l'efficacité et les limitations de la représentation de l'information des signaux par les fonctions de retard de groupe. Nous montrons que la plupart des limitations provient de la nature discrète des manipulations des signaux dans les domaines temporel et fréquentiel. Nous montrons également que les signaux dérivés par les fonctions de retard de groupe par nos algorithmes de reconstruction s'approchent des signaux originaux au fur et à mesure que l'on augmente le nombre d'échantillons dans la TFD. Nous discutons les limitations des fonctions de retard de groupe en termes de position des racines de la transformée en z du signal discret donné. Les fonctions de retard de groupe fournissent une représentation précise de l'information signal tant que les racines ne sont pas trop près du cercle unité dans le plan des z. Les erreurs dans le cas des racines proches due cercle unité sont principalement dues à la reconstruction de la phase à partie de la fonction de retard de groupe.

Keywords. Group delay functions, signal representation, aliasing.

1. Introduction

Representations of signals in the time domain, frequency domain, z-transform and cepstrum are well known in the literature [1]. In [3] it was shown that signal information can also be represented by using two group delay functions, one derived from the magnitude of the Fourier transform and the other from the phase. The objective of this paper is to discuss the adequacy of representation of signals through group delay functions.

Group delay functions have some interesting properties like additive and high resolution properties [2] which can be exploited for many applications such as design of digital filters and pole-zero modelling [3]. These properties enable effective manipulation of signal data in many signal processing situations, like waveform estimation from an ensemble of noisy measurements [4]. But it is not clear whether the group delay functions and the signal reconstruction algorithms from these functions preserve the complete information of the signal or not. We must know under what conditions loss of information, if any, occurs and how common are these conditions in practice.

It was with this objective that we conducted a systematic study to investigate the accuracy of the group delay representation together with the signal reconstruction algorithms from these functions. In Section 2 we introduce the group delay functions and discuss briefly some important properties of these functions. Implementation of the algorithms for computing the group delay functions and for deriving the signal from these functions requires discrete Fourier transforms (DFTs). We study the nature of errors introduced in the group delay functions by the discrete nature of the signal representation. Experiments designed to bring out these errors are described in Section 3. We first discuss the effect of proximity of zeros of the z-transform of the signal to the unit circle in the z-plane and then the effect of the number of such zeros on the signal reconstructed from the group delay functions. We also show that the number of points chosen in the DFT computation affects the reconstruction error. The consequences of these errors in signal processing are discussed in Section 4.

2. Group delay functions

In this section we briefly review the definitions and some important properties of group delay functions. Algorithms for computing the group delay functions, as well as algorithms for deriving the signal from the group delay functions are given in [3]. For implementing the algorithms, it is necessary to use the DFT, which involves discretizing the frequency variable also. This discretization causes aliasing. The severity of this aliasing in deriving the signal from the group delay functions is considered in Section 3.

2.1. Definitions of group delay functions

For a discrete-time signal $\{x(n)\}$, we define the group delay functions as follows:

Let

$$X(\omega)=|X(\omega)|\,e^{j\theta(\omega)} \tag{1}$$

be the Fourier transform (FT) of the signal $\{x(n)\}$. Then the group delay function is defined as the negative derivative of the unwrapped phase function. That is

$$\mathscr{T}_p(\omega)=\frac{-d\theta_u(\omega)}{d\omega}, \tag{2}$$

where $\theta_u(\omega)$ is the phase function in unwrapped form. We call this the group delay function derived from the FT phase. Similarly we define a group delay function $\mathscr{T}_m(\omega)$ derived from the FT magnitude function $|X(\omega)|$. It can be shown [3] that $\mathscr{T}_m(\omega)$ is the negative derivative of the phase of the unique minimum phase equivalent signal derived from $|X(\omega)|$.

2.2. Properties of group delay functions

We give a summary of the important properties of $\mathscr{T}_p(\omega)$ and $\mathscr{T}_m(\omega)$ which were discussed in detail in [3].

(1) For a *minimum phase signal*,

$$\mathcal{T}_p(\omega) = \mathcal{T}_m(\omega).$$

(2) For a *maximum phase signal*,

$$\mathcal{T}_p(\omega) = -\mathcal{T}_m(\omega).$$

(3) For a *mixed phase signal*,

$$|\mathcal{T}_p(\omega)| \neq |\mathcal{T}_m(\omega)|.$$

(4) *Additive property*: Convolution of signals in the time domain is reflected as summation of their respective group delay functions in the group delay domain, as shown in Fig. 1.

(5) *High resolution property*: The resonance peaks (due to complex conjugate pairs of poles or zeros) of a signal are better resolved in the group delay domain than in the spectral magnitude domain [2]. Furthermore, the signal information is confined to the narrow regions around the pole or zero locations, as shown in Fig. 2.

2.3. Problems due to discretization

In general, discretization and quantization may bring about partial loss of information in the group delay domain. The severity of the information loss depends on the nature of the signal being processed. For instance, the linear phase term, which is the average of $\mathcal{T}_p(\omega)$, cannot be computed accurately by averaging the limited number of samples $\{\mathcal{T}_p(\omega)\}$ when there are large fluctuations in that function. Similarly, aliasing in the cepstral domain contributes errors in $\mathcal{T}_m(\omega)$.

In the next section, we investigate the effect of discretization on the representation of signals

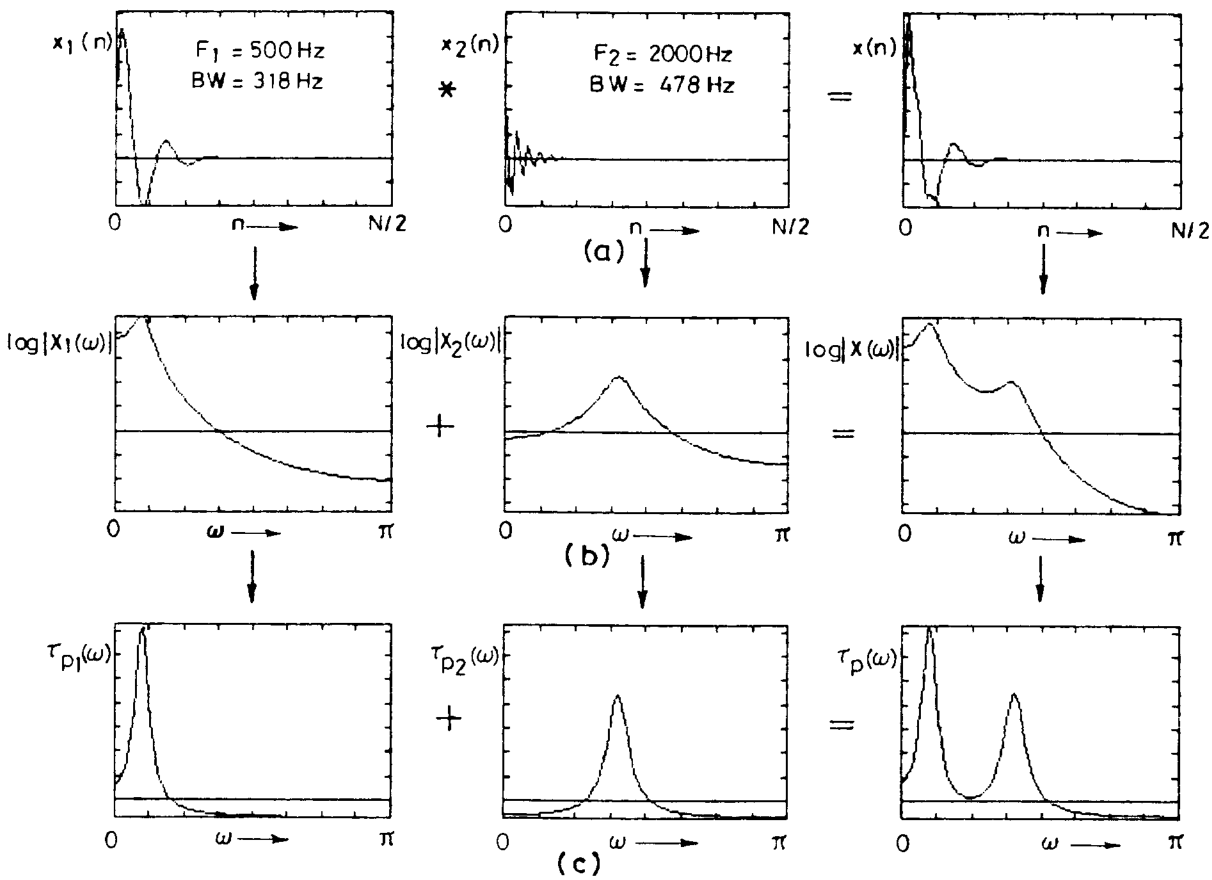

Fig. 1. Illustration of the additive property of the group delay functions: (a) Time domain signals; (b) the corresponding log-magnitude spectra, and (c) group delay functions ($\mathcal{T}_p$).

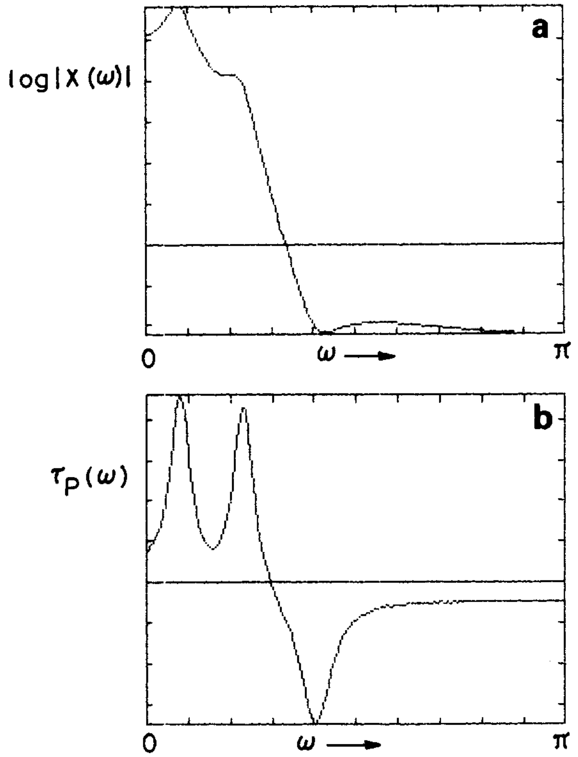

Fig. 2. Illustration of the high resolution property of the group delay functions: (a) Log-magnitude spectrum showing two poles and a zero; (b) the corresponding group delay function ($\mathcal{T}_p$).

through group delay functions and on the reconstructed signals. We identify the signal parameters which may affect the signal reconstruction error. Further, we discuss the nature of dependence of this error on these parameters.

3. Study of the nature of errors in the group delay representation

For any representation to be effective, it is desirable that the complete information in the signal be preserved in the representation. On reconstruction from the group delay functions, if the original signal is obtained without any loss of information, then we can conclude that the group delay representation together with the reconstruction algorithms is accurate.

If continuous time and frequency variables are used throughout, then the errors due to aliasing in the cepstral domain are avoided in the computation of the group delay functions. But digital processing of data necessitates discretization, which may result in partial loss of information. Hence this discretization process of all variables in the computation of the group delay functions and reconstruction algorithms may affect the accuracy of signal representation. Moreover, the accuracy of representation may depend on the characteristics of the signal itself, such as the number of roots and their location in the z-plane with respect to the unit circle.

3.1. Accuracy of signal representation in group delay domain

We will derive an expression for the magnitude group delay function computed from the algorithm given in [3].

Consider a real, causal and finite length sequence $x(n)$ whose z-transform is of the form

$$X(z)=\frac{|A|\prod_{k=1}^{m_i}(1-a_k z^{-1})\prod_{k=1}^{m_o}(1-b_k z)}{\prod_{k=1}^{p_i}(1-c_k z^{-1})}, \quad (3)$$

where $|a_k|$, $|b_k|$ and $|c_k|$ are all less than 1. Here the c_k and a_k correspond to the poles and zeros inside the unit circle, respectively, and the b_k correspond to the zeros outside the unit circle in the z-plane.

The complex cepstrum $\hat{x}(n)$ of $x(n)$ is given by [1],

$$\hat{x}(n)=\begin{cases}\log|A|, & n=0\\ \sum_{k=1}^{p_i}\frac{c_k^n}{n}-\sum_{k=1}^{m_i}\frac{a_k^n}{n}, & n>0\\ \sum_{k=1}^{m_o}\frac{b_k^{-n}}{n}, & n<0\end{cases} \quad (4)$$

The magnitude cepstrum $c(n)$, which is the inverse Fourier transform of the log-magnitude spectrum, is the even part of the complex cepstrum and can be expressed as

$$c(n)=\begin{cases}\log|A|, & n=0\\ \frac{1}{2}\sum_{k=1}^{p_i}\frac{c_k^n}{n}-\frac{1}{2}\sum_{k=1}^{m_i}\frac{a_k^n}{n}-\frac{1}{2}\sum_{k=1}^{m_o}\frac{b_k^n}{n}, & n>0\\ -\frac{1}{2}\sum_{k=1}^{p_i}\frac{c_k^{-n}}{n}+\frac{1}{2}\sum_{k=1}^{m_i}\frac{a_k^{-n}}{n}+\frac{1}{2}\sum_{k=1}^{m_o}\frac{b_k^{-n}}{n}, & n<0\end{cases} \tag{5}$$

The magnitude group delay function is given as

$$\begin{aligned}\mathscr{T}_{\mathrm{m}}(\omega) &= \sum_{n=1}^{\infty} nc(n)\cos n\omega\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{p_i} c_k^n \cos n\omega\\ &\quad -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{m_i} a_k^n \cos n\omega\\ &\quad -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{m_o} b_k^n \cos n\omega.\end{aligned} \tag{6}$$

Equation (6) shows that the magnitude group delay function is the Fourier transform of the summation of the complex exponentials formed by the poles and zeros of the signal. We observe that when any of the roots are close to the unit circle, the corresponding complex exponentials do not decay fast enough with increasing n. Hence, aliasing takes place in the function $nc(n)$, as the DFT size used in the actual implementation is finite. This contributes to errors in the magnitude group delay. We also observe from equation (6) that when the number of roots is increased, the total error, which is the summation of the error contributions of each root, also increases. As the DFT size used in the computation is increased, the severity of aliasing in $\{nc(n)\}$ is reduced and consequently the error in the group delay function would be less. We note here that the magnitude group delay function is more sensitive to aliasing problems than the cepstrum, for the decay of function $nc(n)$ is much slower owing to the multiplication factor n.

The phase group delay function also undergoes errors in the representation of the signal information under a similar set of conditions to those described above. But here these errors manifest themselves as under-sampling of the group delay function. The algorithm for computing phase group delay as given in [1, pp. 495-498] gives accurate values of the group delay at the sample points. But in the reconstruction algorithm from phase group delay, we have to compute the cepstrum, where aliasing occurs owing to the under-sampled phase group delay function. This results in the distortion of the reconstructed signal.

3.2. Experimental results

We conducted some experiments to demonstrate the effects of discretization on group delay representation of signals. The choice of experiments is based on the discussion given in the previous section and our experience with the use of group delay functions over the past several years. Composite signals of the form shown in Fig. 3 are used for these experiments. Each signal is a summation of the windowed impulse response of a 12th order all-pole system (referred to as the basic signal) and its scaled and shifted version (referred to as its echo). We will not consider the effect of windowing in the following discussion although a single-sided Hamming window is used as mentioned above. This signal is represented in the time domain as

$$y(t)=x(t)u(t)+\gamma x(t-t_1)u(t-t_1), \tag{7}$$

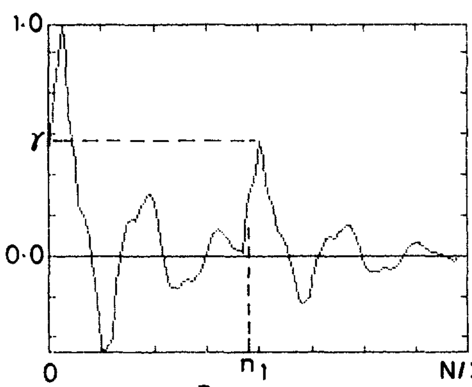

Fig. 3. A typical composite signal used in the experiments.

where $u(t)$ is the unit step function and γ is the ratio of the echo amplitude and the basic signal amplitude. The discrete time version of this signal is given by

$$y(n)=x(n)u(n)+\gamma x(n-n_1)u(n-n_1). \quad (8)$$

Taking the z-transform for the equation (8), we get

$$Y(z)=(1+\gamma z^{-n_1})X(z), \quad (9)$$

where $Y(z)$ and $X(z)$ are the z-transforms of $y(n)$ and $x(n)$ respectively. $X(z)$ contains six pairs of complex conjugate poles located inside the unit circle in z-plane.

From equation (9), we observe that in the z-plane the signal contains twelve poles due to the basic signal and n_1 zeros symmetrically distributed around the origin at a distance of γ from it due to the echo. If $\gamma=1$, the zeros are on the unit circle and when $\gamma<1$ or $\gamma>1$, the zeros are inside or outside the unit circle, respectively.

Experiments were conducted to see the effects of various parameters on the errors resulting from the reconstruction from group delay functions.

The following procedure is followed in all the experiments. The details of the computation algorithms are given in [3].

(1) Form a composite time signal $x(n)$ by taking the summation of a basic signal and its scaled and shifted replica. The basic signal is constructed from the impulse response of a 12th order all-pole system derived from speech data. A single-sided Hamming window is used to limit it to a finite length to ensure that the composite signal sequence is not more than $\frac{1}{2}N$ samples in length. Form an N-point sequence by padding the above sequence with zeros.

(2) Find the log-magnitude and phase sequences by using an N-point DFT of the composite time signal.

(3) Find the magnitude group delay ($\mathcal{T}_m$) from the log-magnitude sequence. Save the scale factor which is equivalent to the average of $\mathcal{T}_m$.

$\mathcal{T}_m$ is computed as follows:

(a) Compute the DFT of the log-magnitude spectrum to obtain the cepstrum $\{c(n)\}$.

(b) Form the even sequence $\{nc(n)\}$.

(c) Compute the IDFT of the sequence $\{nc(n)\}$ to get $\mathcal{T}_m$.

(4) Find the phase group delay ($\mathcal{T}_p$) from the given time domain signal. Save the average value of $\mathcal{T}_p$, which represents the linear phase term.

$\mathcal{T}_p$ is computed as follows:

(a) Find the DFTs $X(k)$ and $Y(k)$ of the sequences $x(n)$ and $nx(n)$ respectively.

(b) Compute the group delay $\mathcal{T}_p$ as

$$\mathcal{T}_p=\frac{X_R(k)Y_R(k)+X_I(k)Y_I(k)}{X_R^2(k)+X_I^2(k)},$$

where the subscripts R and I denote the real and imaginary parts respectively.

(5) Reconstruct log-magnitude from $\mathcal{T}_m$ and restore the scale factor. This can be done by reversing the steps in (3).

(6) Reconstruct phase from $\mathcal{T}_p$ and add the linear phase computed from the average value of $\mathcal{T}_p$.

The phase sequence is reconstructed from $\mathcal{T}_p$ as follows:

(a) Compute the IDFT of $\mathcal{T}_p(k)$ to get the sequence $\{nc(n)\}$.

(b) Form the odd sequence with $\{c(n)\}$.

(c) Compute the DFT of $\{c(n)\}$ to get the phase sequence.

(7) Reconstruct the time domain signal from the reconstructed log-magnitude and the phase.

(8) compare the plots of the original time signal and the reconstructed signal from step (7). Observe the reconstruction error.

The algorithms given in [3] are used for (i) computing magnitude and phase group delays from the spectral magnitude and the time domain signal and (ii) for reconstructing the spectral magnitude and phase from its magnitude and phase group delays. We note that the algorithm for reconstruction of magnitude from $\mathcal{T}_m$ consists of retracing the steps for the algorithm for computing $\mathcal{T}_m$ from log-magnitude, whereas for $\mathcal{T}_p$ the algorithms are entirely different. Hence, the reconstructed log-magnitude does not show any significant error, even though there may be error in $\mathcal{T}_m$. Because of this the reconstruction error in the following experiments is predominantly contributed by error in the phase reconsturcted from $\mathscr{T}_p$. We use the composite signal shown in Fig. 3 in these experiments.

Experiment 1. *Effect of varying the proximity of zeros to the unit circle*

From equation (9) it is seen that by changing the value of γ, which is the ratio of the amplitudes of the echo and the basic signal, we can move the zeros along the radial direction in the z-plane. An experiment was conducted in which γ was varied from 0.5 to 2.0. Fig. 4 shows the superimposed plots of the original and the reconstructed signals for different values of γ. It is observed from these plots that the reconstruction error is negligible for $\gamma = 0.5$, but increases steadily as the zeros approach the unit circle and becomes infinity on the unit circle ($\gamma = 1.0$). As γ is further increased, the zeros fall outside the unit circle and move away from it. In this case it is observed that the reconstruction error decreases and becomes negligibly small when $\gamma = 2.0$. This shows that proximity of roots (in this case zeros) to the unit circle introduces error in the reconstruction irrespective of whether they are inside or outside the unit circle. This may be attributed to the fact that as the zeros approach the unit circle, abrupt changes occur in the spectral phase at the frequencies corresponding to these zeros. Similarly in the spectral magnitude, the valleys due to the zeros become steeper. These changes in the magnitude and phase result in poor sampling of the corresponding group delay functions and hence loss of information in the group delay transformations.

Experiment 2. *Effect of number of zeros*

The effect on the reconstructed signal of varying the number of zeros equidistant from the unit circle was also studied. The delay n_1 in equation (9) is equal to the number of zeros in the z-plane. Hence by varying n_1 in the time domain signal, we can vary the number of zeros. An experiment was conducted in which n_1 was varied from 1 to 20 and the corresponding plots of the original and reconstructed signals are superimposed in Fig. 5. It is seen from these plots that as the number of zeros are increased, the reconstruction error also increases. We may again conclude that this error

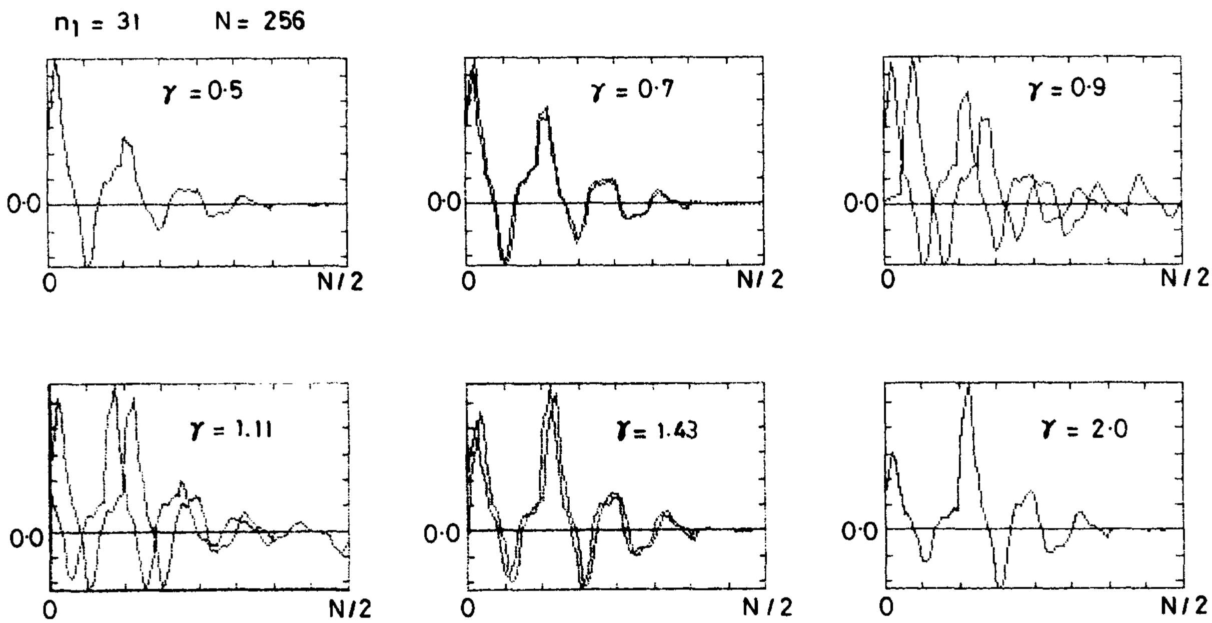

Fig. 4. Comparison of the original and reconstructed signals for different values of echo amplitude (γ).

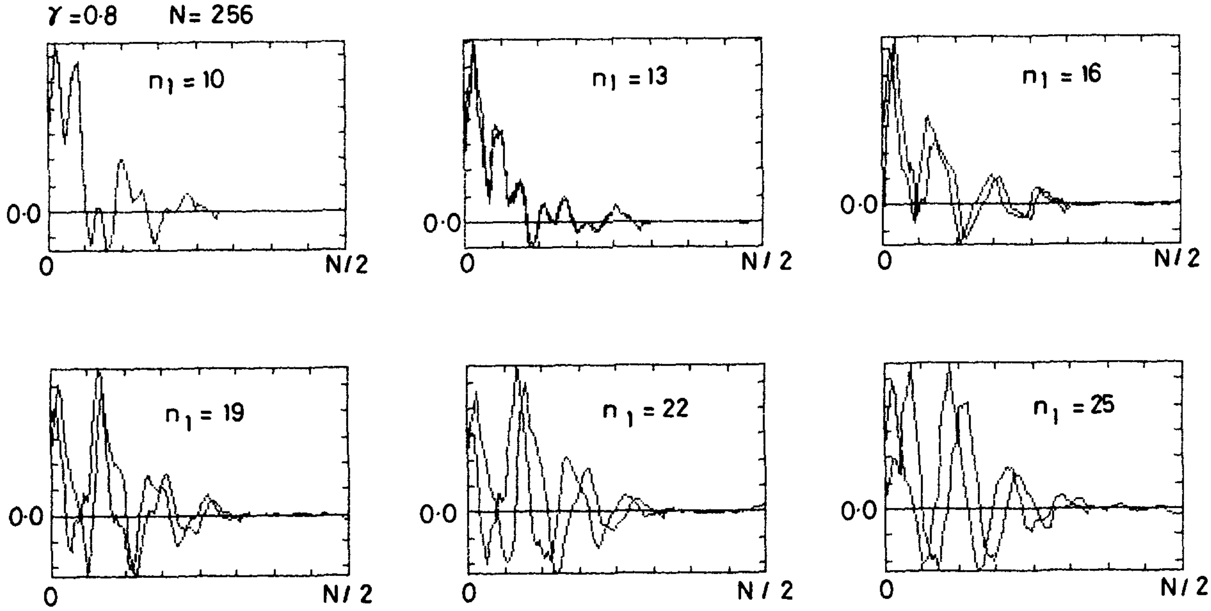

Fig. 5. Comparison of the original and reconstructed signals for different values of the number of zeros (n_1) in the z-plane.

is due to the poor sampling of the group delay functions.

Experiment 3. *Effect of number of* DFT *points*

The time signal length is fixed at 16 samples and zeros are padded to make up the N points. N is changed from 32 to 1024 in steps, and the plots of the original and reconstructed signals for different cases are superimposed in Fig. 6. We observe that the reconstruction error decreases as the number of DFT points are increased. For $N = 1024$, the error is negligibly small. These results also show

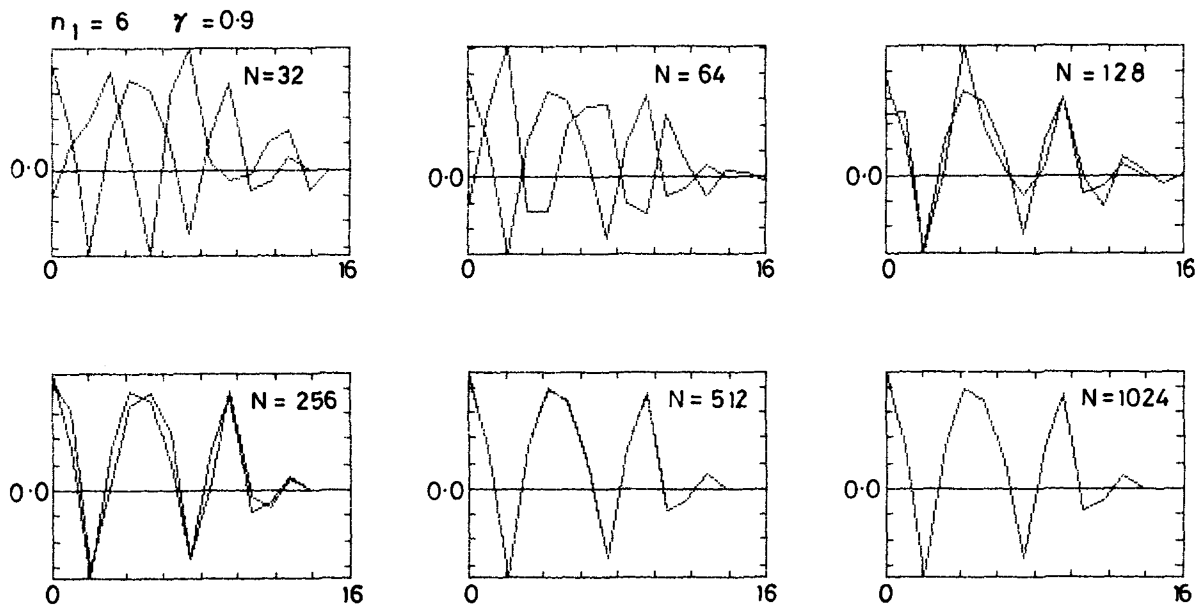

Fig. 6. Comparison of the original and reconstructed signals for different values of the DFT size (N).

that poor sampling of group delay functions causes errors in the reconstruction.

Experiment 4. *Statistical studies*

We have conducted the above experiments on a large set of data to verify whether the results obtained above are consistent. In each case, mean square error between the original and the reconstructed time signals is computed. This error is normalised and expressed in percentage. The data were derived from 60 sets of 12th order all-pole systems. The error plots are shown in Fig. 7. These plots confirm our earlier observations.

4. Conclusions

The studies given in this paper clearly demonstrate the significance of the group delay representation and its limitations in terms of inaccuracies in the signal transformation algorithms. These signal transformation algorithms are accurate if the signals are continuous in all domains. It is interesting to note that the reconstruction errors mentioned in the experiments in the previous section can be made as small as required by taking a sufficiently large number of DFT points, provided there are no roots on the unit circle. This is achieved by zero-padding the given time signal to get the required N-point sequence.

The effect of the characteristics of the signal on the reconstruction error is felt when the number of DFT points chosen is not large. Though phase group delay gives accurate values at the sample points, magnitude group delay is distorted by aliasing in the cepstral domain. The signal transformation routines given in [3] for representation of signals in the group delay domain are accurate only when the number of sample points is sufficiently high in the group delay domain. Hence, whenever signal characteristics, such as the number of roots and/or their proximity to unit circle, result in rapid variation in the spectral magnitude or phase, poor sampling occurs in these domains.

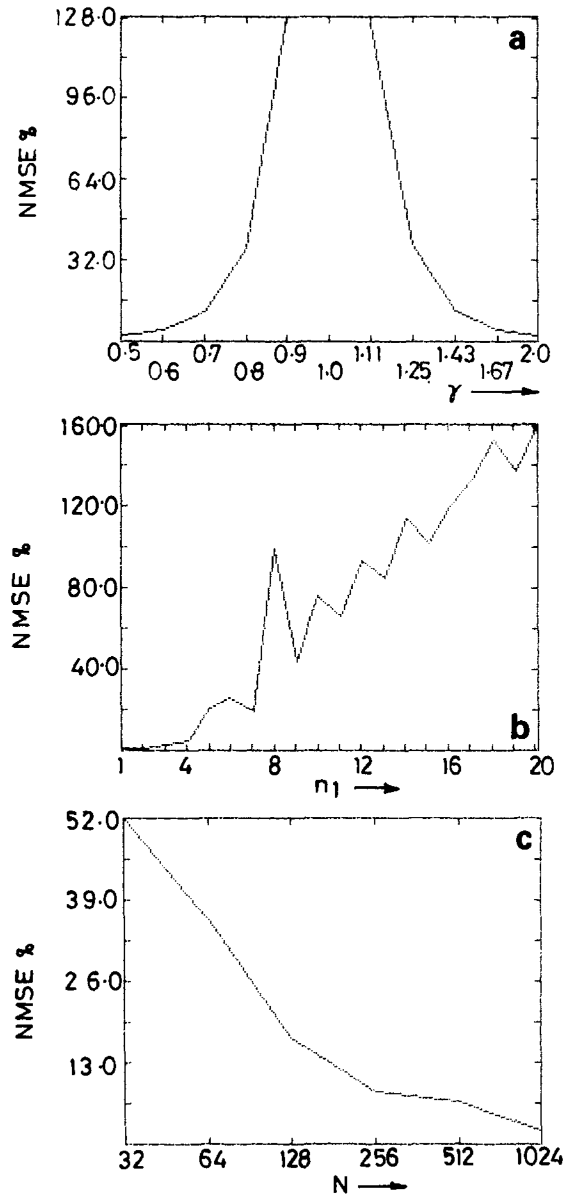

Fig. 7. Reconstruction error averaged over 60 frames of data. The variation in the normalised mean square error (NMSE) with respect to: (a) Echo amplitude (γ); (b) number of zeros (n_1) in the z-plane, and (c) DFT size (N).

We note that adequate sampling based on Nyquist criterion in the time domain does not necessarily result in proper sampling in the group delay domain, because of the derivatives involved in the definition of group delay functions.

The above conclusions are relevant in the context of composite signal decomposition and speech analysis. We will show in another paper how the studies reported in this paper can be effectively used to interpret the results of composite signal decomposition using group delay processing.

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