# ROBUST IMAGE REGISTRATION WITH ILLUMINATION, BLUR AND NOISE VARIATIONS FOR SUPER-RESOLUTION

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**ABSTRACT**

Super-resolution reconstruction algorithms assume the availability of exact registration and blur parameters. Inaccurate estimation of these parameters adversely affects the quality of the reconstructed image. However, traditional approaches for image registration are either sensitive to image degradations such as variations in blur, illumination and noise, or are limited in the class of image transformations that can be estimated. We propose an accurate registration algorithm that uses the local phase information, which is robust to the above degradations. We derive the theoretical error rate of the estimates in presence of non-ideal band-pass behavior of the filter and show that the error converges to zero over iterations. We also show the invariance of local phase to a class of blur kernels. Experimental results on images taken under varying conditions clearly demonstrates the robustness of our approach.

***Index Terms*—** Registration, Super-Resolution, Local Phase.

## 1. INTRODUCTION

Generating high-resolution images from multiple low-resolution, degraded images has a variety of applications in space imaging, medical imaging, commercial videography, etc. Any Super-Resolution(SR) algorithm assumes accurate blur and registration parameters. Most of the existing registration algorithms perform well in presence of uniform illumination across frames as well as limited and uniform blur and noise. However, these conditions are frequently violated in real-world imaging, where specular surfaces, close light sources, small sensors and lenses create large variations in illumination, noise, and blur within the scene. Interestingly, these are the exact situations, where one would like to employ super-resolution algorithms.

The primary factor that controls the quality of the super-resolved image is the accuracy of registration of the low resolution frames. Park *et al.* [1] has shown by example that small error in registration can considerably effect the super-resolution results. Most multi-image super resolution algorithms assume that the exact registration parameters between the constituent frames are known. However, as mentioned before, the image artifacts can affect the accuracy of estimation of these parameters. Typically, two characteristics of registration have been considered in the past:

*(a) Accuracy:* Super-resolution algorithms require extremely precise alignment of the low-resolution frames; accurate to the order of a tenth of a pixel. However, most of these algorithms tend to be sensitive to illumination, blur variations and noise. Examples of such algorithms include RANSAC [2] and gradient descent based approaches which minimize the squared error of pixel intensities [3]. Robinson *et al.* [4] also proposed a statistically optimal registration technique based on intensity values. The registration parameters in such approaches converge to incorrect values under image artifacts, specifically, non-uniform illumination. Although the RANSAC-based registration is robust in presence of outliers, its performance is restricted by the reliability of feature detectors, which is considerably affected by many image artifacts.

*(b) Robustness:* Registration algorithms that are robust to image artifacts are available, and have been used in applications such as registering multi-modal medical and space images. The primary concern of these algorithms is to address large variations in the image, while being moderately accurate. However, the accuracy of such approaches is too low to be considered for SR applications. Approaches that use frequency domain processing to compute the registration parameters are relatively stable under various image artifacts. However, they are limited in the class of transformations that can be estimated between two images [5]. Further reviews of the registration algorithms can be seen in paper by Park *et al.* [1].

The image formation process used in Super Resolution (SR) reconstruction is given by a linear system,

$$\mathbf{y}_k = \mathbf{L}_k\mathbf{D}_k\mathbf{B}_k\mathbf{F}_k\mathbf{x} + \mathbf{n}_k, \quad (1)$$

where $1 \leq k \leq n$, $\mathbf{x}$ and $\mathbf{y}_k$ are the high and low resolution images respectively. The geometric transformation parameters are captured by the matrix, $\mathbf{F}_k$, which are estimated by the registration algorithms, $\mathbf{B}_k$ is the blurring matrix, and the illumination variations are taken care by the diagonal matrix $\mathbf{L}_k$. However, current algorithms deal with the illumination variation, only at the SR phase [6], and assumes accurate registration. Solutions that deal with registration error by treating it as noise [7] during the SR phase, and a combined optimization of SR and registration [8] have been tried. However, with larger amounts of registration error and outliers, the results will degrade fast, or will not converge in more complex optimizations.

In this paper, we explore at an alternate solution to the problem of robustness in the registration step of a SR algorithm. We formulate the registration as optimization of the local phase alignment at various spatial frequencies and directions. The local phase in an image has been used for problems such as estimation of stereo disparity [9], and optical flow field estimation [10]. We extend its scope to estimate accurate registration parameters and use it for computing super-resolved images. In this paper, we: 1) propose a registration framework using local-phase, which is known to be robust to noise and illumination parameters, 2) derive the theoretical error rate of the approach introduced by limitations of Finite Impulse Response (FIR) filters and show that the algorithm converges to the actual registration parameters, 3) show that the algorithm is not sensitive to a large class of blur kernel functions; and 4) present experimental results of SR reconstruction, that demonstrates the advantages of this approach as compared to other popular techniques.

## 2. LOCAL PHASE BASED REGISTRATION

Accurate registration can be achieved with the exact knowledge of degradation parameters such as blur and non-uniform illumination. However, in practice, this information is rarely available. We overcome this problem by using local phase to estimate registration parameters. Local phase is robust towards noise and smoothly varying illumination [11]. We prove the invariability of local phase information to a class of blur kernels. Due to these characteristics, our registration algorithm can easily by-pass these image artifacts, which are difficult to estimate accurately.

Local phase can be computed using any FIR band pass filter. The phase, as opposed to magnitude of the filter response, is robust [11] to Gaussian white noise. Existing registration algorithms routinely achieve upto pixel-level accuracies. However, for finer registration, features should be calculated with sub-pixel accuracy, even under

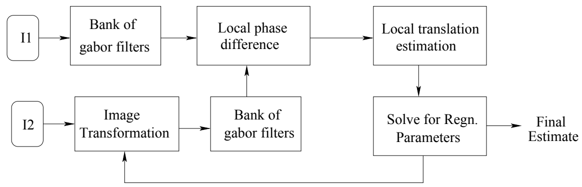



**Fig. 1**. Block diagram showing different steps of the algorithm.

various image artifacts. Local phase based registration can achieve this without explicit signal reconstruction, sub-pixel feature detection or correspondence computation. Local phase has been effectively used to solve similar problems such as stereo disparity computation [9] and optical flow [10] for noisy images.

Gabor filters are popular band pass filters as they achieve the theoretical minimum product of spatial width and bandwidth, desirable for better localization and accurate phase computation, respectively. Mathematically, a Gabor filter is a multiplication of a complex harmonic function with a Gaussian envelope [12],

$$g(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} e^{j(\omega_x x+\omega_y y)}, \quad (2)$$

where, $(\omega_x, \omega_y)$ is the angular frequency of the filter, $\sigma_x$ and $\sigma_y$ controls the spatial width of the filter, and $j$ is $\sqrt{-1}$. Local phase is computed at angular frequency $(\omega_x, \omega_y)$ at each pixel location by convolving the image with Gabor wavelet $g(x, y)$. The argument of the complex output is local phase. Phase difference is computed by taking the difference of phase values at each location of the image pair at the given angular frequency.

**Confidence measurements:** Errors could be introduced in phase difference computation due to noise and the absence of the local frequencies with which the images are convolved. Sanger [9] has described the degree of match in the amplitude values as a confidence measure. The value of confidence is high if the amplitudes of the Gabor filter response at $(x, y)$ in both the images are close. In addition, if the amplitude falls below a particular threshold, the confidence value is set to zero. Let $|s_1|$ and $|s_2|$ be the amplitudes of the Gabor filter response. The confidence value is computed as:

$$r = min\left[\frac{|s_1|}{|s_2|}, \frac{|s_2|}{|s_1|}\right] \quad (3)$$

### 2.1. Registration Algorithm

Our local phase based registration algorithm is robust to noise, illumination, blur and sub-sampling. We convolve the partially overlapping images with Gabor filters at multiple frequency pairs. The idea of convolving with multiple frequencies is that in case the same frequency is not present at both the corresponding location then that observation could be pruned. The local translation parameters are computed at each spatial location from the robust phase difference estimation. An overdetermined system of equation is formed and from these estimates the registration parameters are computed. The transformation parameters are updated iteratively so that errors due to uncertainty in the frequency estimation of the band-pass filter is minimized. Moreover, in any small 2D filter window the corresponding point locations should lie within the cycle of the sinusoid. This condition should hold true at most of the image locations, for our algorithm to converge.

**2D Local Translation:** In the 1D case, the shift between two sinusoids of the same frequency is estimated by measuring the phase difference at the same spatial location and then dividing it by the frequency of the signal. The computation of translation components can be formulated on the basis of Fourier Shift theorem, according to which, a shift of $\Delta x$ in the spatial domain would produce a phase difference of $\Delta x\omega_x$ at $\omega_x$. This is extended in 2D as, a shift of $(\Delta x, \Delta y)$ in the spatial domain would produce a phase difference of $(\Delta x\omega_x + \Delta y\omega_y)$. By computing the phase difference at least at two different angular frequency pair we can estimate $(\Delta x, \Delta y)$. We choose various combinations of $(\omega_x, 0)$, $(0, \omega_y)$ to compute $\Delta x$ and $\Delta y$ respectively. Other combinations of angular frequencies are avoided because solving equations in two variable is very sensitive if both the angular frequencies are very close and the error can go up with increasing noise. The inclusion of confidence parameters in the final computation of variables is straightforward if we restrict the angular frequencies to these two classes. Gabor filter will act as a low-pass filter in orthogonal directions and the effect of noise is reduced. The phase difference is computed at multiple frequency pairs in each dimension and is combined by taking the average of estimates weighted by the confidence values. A pixel is removed from consideration for computing the registration parameters if there is not sufficient response of Gabor filters at all frequencies. This approach is correspondenceless. The local translation parameters thus estimated are accurate at sub-pixel level and computation from multiple frequencies make the estimation robust.

**Frequency Selection at each Iteration:** From the phase of the convolution product, as given by equation 5, the observation is that for a constant spatial window width, local phase is more accurate at higher frequency. But at higher frequency the domain of convergence decreases. The frequency of the band-pass filter is changed from low to high as the algorithm converges. At each iteration, various angular frequencies of the Gabor filter are selected such that they are close. In any iteration, no merging of low and high frequencies is done to compute local translation parameters.

**Registration Parameters:** Local translation parameters thus computed at various spatial image locations can be thought as point correspondences with high accuracy. Given many such corresponding pairs, the image transformation parameters can be estimated by solving an overdetermined system of equation. This framework allows to calculate any type of registration. For our experiments, we limit the class of registration algorithms to that of planar views related by affine transformation. This is because most of the partial overlap, e.g. images captured from a video sequence can be approximated by affine transformations. At each location, we estimate the translation parameters, which is related to the correspondence of a point $(x, y)$ in one image with $(x', y')$ in the other. We form an overdetermined system of equations in image transformation parameters and estimate the accurate registration parameters.

The local translation parameters, calculated at each spatial location, are approximately correct. This is because in a small window points need not be related by pure translation. Moreover, the two points need not lie within the cycle of the signal. However, over iterations, as the corresponding points come closer, the effect due to these assumptions would be negligible. We iteratively update the transformation parameters till convergence.

## 3. CONVERGENCE, ERROR, ROBUSTNESS ANALYSIS

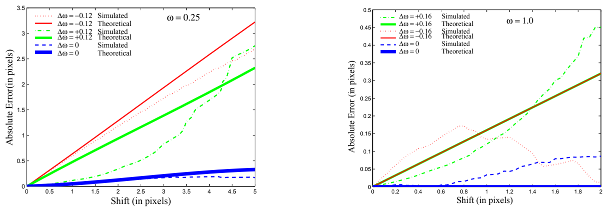
Noise, blur, illumination effects the accuracy of any registration algorithm. We analyze the performance of local phase under these artifacts. Analysis is performed for 1D signals but extensions to 2D is simple and the analysis is same. We consider a 1D Gabor filter, $g(x)$, with angular frequency $\omega_0$ and a sinusoidal signal given by,

$$i(x) = \cos((\omega_0 + \Delta\omega)(x + t)), \quad (4)$$

where $\Delta\omega$ captures the non-ideal band pass behavior of the filter and at cut-off frequency it is the half band-width of the filter, $t$ is the initial shift. The convergence and error bound is computed by analyzing the sinusoids at the cut-off frequencies of the Gabor filter.

**Non-Ideal Band Pass Behaviour of the Gabor Filter:** Gabor Filter has the minimal constant product of spatial width and bandwidth so there is a trade-off in selecting their sizes. Smaller spatial width does help in localization but at the cost of non-zero bandwidth. We convolve the Gabor filter, $g(x)$, with the sinusoid (eq. 4). The phase of the convolution product is (see appendix A.1),

$$\phi_t(x) = \tan^{-1}\left[\tan((\omega_0+\Delta\omega)(t+x))\left(\frac{1-e^{-2(\omega_0^2+\omega_0\Delta\omega)\sigma^2}}{1+e^{-2(\omega_0^2+\omega_0\Delta\omega)\sigma^2}}\right)\right], \quad (5)$$




(a) $\omega_0 = 0.25, \sigma = 5$ (b) $\omega_0 = 1.0, \sigma = 4$

**Fig. 2**. Error in calculation of shift due to non-ideal bandpass filter for various shift values. Solid lines show the theoretical, dotted lines shows the simulated behavior.

At infinite width or at very high frequency the local phase computed is accurate. To show the convergence of phase based registration algorithm we only show that the local translation parameters are computed accurately over iterations at cut-off frequencies of the Gabor filter (cut-off frequency is calculated using Heisenberg's uncertainty principle [12]). The error is calculated for each value of shift as the absolute difference between the actual shift and the shift computed using $(d = (\phi_2 - \phi_1)/\omega)$ ($\phi_2$ and $\phi_1$ are the local phase computed using equation 5 and we assume that only one sinusoid is present). This theoretical error rate is plotted against the simulated convolutions where the sinusoid is quantized and sampled on a grid after its magnitude is scaled by 128. From the error graphs (Figure 2), we conclude that the error drops to zero over iterations. Note that even for ideal band-pass filter the error is not zero at low frequency.

**Blur:** Given a sinusoid, $i(x)$ (eq 4), and an even and real blur kernel, $b(x)$, the local phase is independent of all parameter of blur kernel but the magnitude is scaled (see Appendix A.2). Two images can be compared by local phase information in presence of blur. However, higher frequency information is degraded because of sampling on a grid. The blur parameters varies due to variation in depth and for a planar scene the variation is smooth. It can safely be assumed that in a small window the blur parameters are constant.

**Illumination, Noise and Quantization Errors:** Illumination change, in image space, is the multiplication of pixel value by another value. Smooth illumination can be modeled by the multiplication of a constant in a window. The phase information computed at these two locations will remain unchanged as compared to the magnitude of the signal, which will be scaled by the illumination constant. Fleet and Jepson [11] has shown that the phase is more robust for image matching than the amplitude of the filter response in presence of noise. Quantization errors can also be modeled as noise. The quantization error results from the mapping of irradiance field onto digital sensors. For band-limited noise, the error in the estimation is reduced by considering the phase output of those filters that do not allow those frequencies to pass through. This is done by assigning low scores to those phase difference estimates where there is a significant amplitude mismatch in both the signals detected.

## 4. EXPERIMENTS AND RESULTS

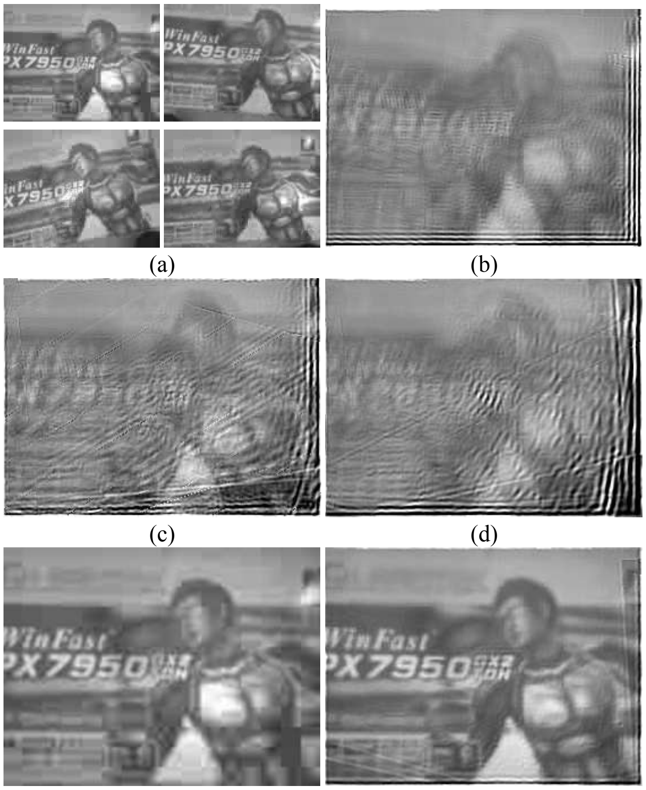
Experiments have been performed on synthetic low-resolution frames and on more challenging real-life images captured using a mobile phone camera (Nokia 3320). The results of our algorithm has been compared with registration algorithm which is based on minimization of mean squared intensity differences [3](IM) using gradient descent and RANSAC [2]. Intensity minimization algorithm has been chosen because it is very accurate in presence of noise and widely used. RANSAC is robust in presence of outliers. We use the algorithm mentioned in [13] for super-resolution reconstruction without any regularization term. For synthetic data-sets the registration algorithms have been compared by using absolute Mean Shift Error ($\epsilon_{ms}$). It is absolute mean of the amount of displacement required to place a point in one image (after applying the calculated transformation) onto the reference image for each pixel. Super-Resolved image has been compared with the ground truth using Root Mean Square Reconstruction Error ($\epsilon_{re}$) on intensity values.

We generated three different kind of data sets of low resolution(LR) frames, which are related by affine transform, from a single high resolution(HR) frame. In first data-set Gaussian noise of various levels were added all under same Gaussian blur of window size 4 and variance 2. In second data set, smooth spatially varying blur with window size smoothly varying between 5 and 9 in different directions was added in each LR frame. In third data-set noise having variance 3 and uniform Gaussian blur was added. Non-uniform illumination was synthetically generated which degrades radially from a randomly selected point source for each of the LR frames. Table 1 summarizes the result for noisy data-set. For second data-set we got the absolute mean shift error as 0.379, 0.674, 0.285 for IM, RANSAC and our algorithm respectively. For third data set, where each low resolution frame has different kind of illumination variations we got the registration error as 5.849, 1.391 and 0.210 respectively. Images were magnified by a factor of 1.8 in all the cases. More challenging real-world video was taken using a mobile phone camera (Nokia 3320). (Video were taken such that all the LR frames are related by affine transformation only by keeping the camera only in one plane). Different part of the scene was illuminated during the video capture by using a flashlight. 8 frames(each had 128 × 96) were selected out of the video for scene super-resolution. Compression artifacts are clearly visible in all LR frames. Fig. 3 summarizes the result. The magnification factor was 2.2. Our algorithm has significantly performed better than the other image registration algorithms. The main reason is lack of features in such a small and heavily degraded image and strong illumination artifacts.

| | Bicubic | Ideal | RANSAC | | IM | | Proposed | |
|---|---|---|---|---|---|---|---|---|
| $\sigma$ | $\epsilon_{re}$ | $\epsilon_{re}$ | $\epsilon_{ms}$ | $\epsilon_{re}$ | $\epsilon_{ms}$ | $\epsilon_{re}$ | $\epsilon_{ms}$ | $\epsilon_{re}$ |
| 1 | 18.28 | 6.87 | 0.56 | 14.71 | 0.189 | 9.23 | 0.195 | 9.39 |
| 2 | 18.34 | 7.28 | 0.77 | 15.57 | 0.189 | 9.83 | 0.201 | 10.07 |
| 3 | 18.43 | 8.08 | 0.79 | 16.10 | 0.187 | 10.21 | 0.216 | 10.44 |
| 4 | 18.55 | 8.39 | 0.83 | 17.11 | 0.187 | 10.72 | 0.221 | 11.11 |
| 5 | 18.72 | 8.97 | 0.65 | 17.84 | 0.186 | 11.17 | 0.223 | 11.67 |
| 6 | 18.91 | 9.45 | 0.92 | 19.01 | 0.191 | 11.87 | 0.223 | 12.27 |

**Table 1**. Comparison of the proposed scheme with other image registration algorithms under Gaussian white noise (with 0 mean and standard devation, $\sigma$, from 1-6). (Ideal) denotes the error when actual registration parameters are given as input for SR reconstruction.

**Discussions:** Our algorithm performs better than RANSAC for all 3 data sets and is comparable to intensity minimization algorithm (IM) under Gaussian white noise. The optimization framework for IM can be shown to be independent of Gaussian white noise, and hence its performance is marginally better than the proposed one. However, our algorithm clearly outperforms IM in presence of non-uniform illumination and non-uniform blur. SR applied on the images taken from the video of the mobile phone camera shows the robustness and practical applicability of our algorithm. The compared algorithm fails miserably in such cases due to the lack of feature points, compression artifacts, small size of the image, and high level of degradations. Moreover, our algorithm is correspondence-less and does not require the calculation of feature points. Experiments were performed for images related by an affine transformation. However, our algorithm is easily extensible to a general class of image transformations. The computation of local phase requires a minimum amount of texture in the image. As we need not compute exact feature locations, the absolute intensity values need not be preserved across the image, and hence can deal with varying blur and illumination. Existing transformed domain techniques are more robust to these artifacts. But they solve a very small class of image transformations. Our algorithm can estimate the local translation accurately, given that the corresponding points lie within the cycle of the signal (8-10 pixels apart in practice). By quick registration using any existing image registration algorithm we can overcome this limitation.



(a) (b) (c) (d) (e) (f)

**Fig. 3**. (a) various LR frames (4 out of 8 used) of a video with spatially varying illumination captured using a mobile phone camera; SR reconstruction results using different registration algorithms (b) intensity minimization (c) RANSAC (sift point detector) (d) RANSAC (Harris point detector) (e) single frame bicubic (f) phase-based method; compression artifacts is absent in (f) and the center text is more clear

## 5. CONCLUSIONS

We proposed an algorithm for image registration, which is robust in presence of noise, non-uniform blur and illumination. We have shown that our algorithm based on local phase is independent of blur and illumination artifacts. Our approach is also correspondence-less, and hence there is no need of calculating features, explicitly. We have proven the convergence of the algorithm, even when it is impossible to identify the exact frequency of the underlying signal. Our algorithm is extensible to any general class of image registration which is not the case with other transformed domain approaches though both provides similar robustness.

## 6. REFERENCES


[1] Sung C. Park, Min K. Park, and Moon G. Kang, "Super-resolution image reconstruction: a technical overview," *IEEE, Signal Processing Magazine*, vol. 20, no. 3, pp. 21–36, 2003.

[2] D. Capel and A. Zisserman, "Computer vision applied to super resolution," *IEEE S Pr Mag*, vol. 20, no. 3, pp. 75–86, May 2003.

[3] Luis Ibanez, Will Schroeder, Lydia Ng, and Josh Cates, *The ITK Soft. Guide*, Insight Soft. Consortium, 2005, http://www.itk.org.

[4] Dirk Robinson, Sina Farsiu, and Peyman Milanfar, "Optimal registration of aliased images using variable projection with applications to super-resolution," *The Computer Journal*, April 2007.

[5] P. Vandewalle, S. Süsstrunk, and M. Vetterli, "A Frequency Domain Approach to Registration of Aliased Images with Application to Super-Resolution," *EURASIP Journal on Applied Signal Processing*, vol. 2006, pp. Article ID 71459, 14 pages, 2006.

[6] Wen-Yi Zhao, "Super-resolution with significant illumination change," *In Proc. of ICIP*, vol. 3, pp. 1771–1774, 2004.

[7] Wen-Yi Zhao and Harpreet S. Sawhney, "Is super-resolution with optical flow feasible?," in *ECCV (1)*, 2002, pp. 599–613.

[8] Russell C. Hardie, Kenneth J. Barnard, and Ernest E. Armstrong, "Joint map registration and high-resolution image estimation using a sequence of undersampled images.," *IEEE Transactions on Image Processing*, vol. 6, no. 12, pp. 1621–1633, 1997.

[9] T. D. Sanger, "Stereo disparity computation using Gabor filters," *Biological Cybernetics*, vol. 59, pp. 405–418, 1988.

[10] T. Gautama and M.M Van Hulle, "A phase-based approach to the estimation of the optical flow field using spatial filtering," *IEEE Trans. Neural Nets.*, vol. 13, no. 5, pp. 1127–1136, 2002.

[11] D. J. Fleet and A. D. Jepson, "Stability of phase information," *PAMI*, vol. 15, no. 12, pp. 1253–1268, 1993.

[12] Joni-Kristian Kämäräinen, *Feature Extraction Using Gabor Filters*, Ph.D. thesis, Lappeenranta Univ. of Technology, 2003.

[13] S. Farsiu, D. Robinson, M. Elad, and P. Milanfar, "Robust shift and add approach to super-resolution," *In SPIE Conf. on Appl. of Digital Signal and Image Processing*, pp. 121–130, Aug. 2003.

[14] Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, *Signals & systems (2nd ed.)*, Prentice-Hall, Inc., 1996.


### A1: Derivation of Equation 5

We rewrite $i(x)$ in Euler form as,

$$i(x) = (e^{j(\omega_0+\Delta\omega)x}e^{j(\omega_0+\Delta\omega)t} + e^{-j(\omega_0+\Delta\omega)x}e^{-j(\omega_0+\Delta\omega)t})/2$$

The convolution product with the Gabor filter is

$$r(x) = i(x) * g(x)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}(e^{j(\omega_0+\Delta\omega)x'}e^{j(\omega_0+\Delta\omega)t} + e^{-j(\omega_0+\Delta\omega)x'}e^{-j(\omega_0+\Delta\omega)t})$$

$$.e^{-\frac{(x-x')^2}{2\sigma^2}}e^{j\omega_0(x-x')}dx'$$

After simplifying the above equation we have,

$$r(x) = \sigma\sqrt{\pi/2}\left(e^{j(\omega_0+\Delta\omega)(x+t)}e^{-\frac{1}{2}(\Delta\omega\sigma)^2}\right)$$

$$+ \sigma\sqrt{\pi/2}\left(e^{-j(\omega_0+\Delta\omega)(x+t)}e^{-\frac{1}{2}(2\omega_0+\Delta\omega)^2\sigma^2}\right)$$

Let $\theta=(\omega_0+\Delta\omega)(x+t)$, $A=e^{-\frac{1}{2}(\Delta\omega\sigma)^2}$ and $B=e^{-\frac{1}{2}(2\omega_0+\Delta\omega)^2\sigma^2}$

$$r(x) = \sigma\sqrt{\pi/2}\left(Ae^{j\theta} + Be^{-j\theta}\right)$$

$$= \sigma\sqrt{\pi/2}\,((A+B)\cos\theta + j(A-B)\sin\theta)$$

Phase of $r(x)$, after substitution and simplification is equation 5.

### A2: Blur Invariance

First we derive the convolution of $b(x)$, a blur kernel which is real and even, and $i(x)$, a sinusoid (equation 4). For simplicity let $\omega_p = \omega_0 + \Delta\omega$. Let $B(\omega)$ is the Fourier transformation of $b(x)$ and $I(\omega)$ is the Fourier transformation of $i(x)$ which is [14]:

$$I(\omega) = e^{j\omega t}(\pi\delta(w-\omega_p) + \pi\delta(w+\omega_p))$$

Using the fact that if $b(x)$ is real and even, then $B(\omega)$ will be real and even [14] and the convolution in spatial domain is equivalent to multiplication in frequency domain. So,

$$R(\omega) = B(\omega)I(\omega)$$

$$= e^{j\omega t}(\pi B(\omega_p)\delta(w-\omega_p) + B(-\omega_p)\pi\delta(w+\omega_p+))$$

$$= B(\omega_p)e^{j\omega t}(\pi\delta(w-\omega_p) + \pi\delta(w+\omega_p))$$

Note that $\omega_p$ is a constant. Taking the inverse Fourier transformation,

$$r(x) = B(\omega_0+\Delta\omega)\cos((\omega_0+\Delta\omega)(x+t))$$

Convolution of $r(x)$ with the Gabor filter is simple by following the steps in App. A.1 from here, we note that the phase information remains invariant, while the amplitude is multiplied by $B(\omega_0+\Delta\omega)$.