

VERTEX MAGIC TOTAL LABELINGS OF COMPLETE GRAPHS ¹

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Abstract

The study of graph labeling has focussed on finding classes of graphs which admits a particular type of labeling. In this paper we consider a particular class of graph which admits a vertex magic total labeling. The class we considered here is the class of complete graphs, K_n . A vertex magic labeling of a graph is a bijection which maps the vertices V and edges E to the integers from $1, 2, 3, \dots, |V| + |E|$, such that the sum of label on a vertex and labels of its incident edges is a constant, independent of the choice of the vertex. We make use of technique of generating magic squares and ideas from graph factorization in our construction.

Keywords: complete graphs, vertex magic total labeling, factorization.

2000 Mathematics Subject Classification: 05C78 – Graph labelling.

1 Introduction

Let $G = (V, E)$ be a graph which is finite, simple and undirected. The graph G has a vertex set $V = V(G)$ and edge set $E = E(G)$. We denote an $e = |E|$ and $v = |V|$. A general reference for graph theoretic notations is [5]. In this paper we deal only with complete graphs. The labeling of a graph is a map that takes graph elements V or E or $V \cup E$ to numbers (usually positive or non-negative integers). In this paper the domain is set of all vertices and edges giving rise to a total labeling. The most complete recent survey of graph labelings[1]. The notation of a vertex magic total labelling was introduced in [3,4].

Vertex magic total labeling is an assignment of integers from $1, 2, 3, \dots, v + e$ to the vertices and edges of G such that the sum of vertex label and labels of the edges incident at that vertex is equal to a constant. A more formal way of defining a vertex magic total labelling is a bijection f from $V \cup E$ to the set of integers $1, 2, 3, \dots, v + e$ such that

¹This work is supported in part by the Department of Science and Technology, Govt. of India under Project No. SR/S4/MS:406/06

there is a constant k for every vertex v ,

$$f(v) + \sum_{w|vw \in E} f(v, w) = k.$$

Miller, Macdougall, Slamun, and Wallis [4] gave a vertex magic total labeling of complete graphs K_n for $n \equiv 2 \pmod{4}$ by making use of vertex magic total labeling of $K_{\frac{n}{2}}$, where $n/2$ is odd. Lin and Miller [3] gave a vertex magic total labeling of complete graphs K_n for $n \equiv 4 \pmod{8}$ by making use of vertex magic total labeling of $K_{\frac{n}{4}}$, where $n/4$ is odd. In this paper we prove that *every complete graphs K_n , has a vertex magic total labeling.*

In this paper, we propose constructions for arriving at a vertex magic total labeling for all complete graphs. Our constructions are very simple compared to the ones proposed in the literature [3] which uses mutually orthogonal latin squares. Moreover, while our method is similar to that [7] in some respects, the magic constant obtained by our method is smaller than the constant reported by [3, 7]. Further, the intuitive representation scheme makes the proposed method easy to understand.

In this paper, we use ideas from *twin-factorizations* [6] to build the desired labelings using magic squares which are very simple to build and understand. In this direction, we first build a vertex magic total labeling of K_n when n is odd (cf. Theorem 2.1). This is then used to build a vertex magic total labeling of K_n when n is even by treating it as three cases: $n \equiv 2 \pmod{4}$, $n \equiv 4 \pmod{8}$, and $n \equiv 0 \pmod{8}$. The labeling for even n is obtained by representing K_n as a union of 3 graphs: 2 $K_{n/2}$ and a $K_{n/2, n/2}$. However, care must be taken so that the labels for these smaller graphs can be composed to get the labels for K_n . This requires that the set of labels 1 through $n + n(n-1)/2$ is partitioned so that these partitions can be used to label the 3 graphs independently and when combined result in a vertex magic total labeling of K_n .

A Note on Representation To help us in visualizing the labels of the edges and the vertices, we use the following matrix representation for the labels. We represent the vertex magic total labeling for K_n as an $n \times n$ matrix in which the entries of the first row were used to label the vertices of the graph. The remaining entries of the matrix were used to label the edges of graph. The i th column of the matrix contains the labels of all the edges incident at vertex i , in rows 2 through n in the order $(v_i, v_1), (v_i, v_2), \dots, (v_i, v_{i-1}), (v_i, v_{i+1}), \dots, (v_i, v_m)$. The final matrix takes the general form as shown in Figure 1 below. Notice that in this matrix, each column sum shall be constant, called the *magic constant*.

The rest of the paper is organized as follows. In Section 2, we show the construction of the labeling for K_n , when n is odd. Section 3 shows the constructions for the case when n is even. We end the paper with some concluding remarks in Section 4.

| | | | | | | |
|--------------|--------------|--------------|--------------|--------------|----------|--------------------|
| v_1 | v_2 | v_3 | v_4 | v_5 | \cdots | v_n |
| (v_1, v_2) | (v_2, v_1) | (v_3, v_1) | (v_4, v_1) | (v_5, v_1) | \cdots | (v_n, v_1) |
| (v_1, v_3) | (v_2, v_3) | (v_3, v_2) | (v_4, v_2) | (v_5, v_2) | \cdots | (v_n, v_2) |
| (v_1, v_4) | (v_2, v_4) | (v_3, v_4) | (v_4, v_3) | (v_5, v_3) | \cdots | (v_n, v_3) |
| (v_1, v_5) | (v_2, v_5) | (v_3, v_5) | (v_4, v_5) | (v_5, v_4) | \cdots | (v_n, v_4) |
| \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot |
| \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot |
| (v_1, v_n) | (v_2, v_n) | (v_3, v_n) | (v_4, v_n) | (v_5, v_n) | \cdots | $(v_n, v_{(n-1)})$ |

Figure 1: The matrix of labels for K_n . The first row has the vertex labels and the i th column entries in rows 2 to $n - 1$ have the edge labels for v_i , for $1 \leq i \leq n$.

2 Vertex magic total labeling for K_n where n is odd.

Our construction of vertex magic total labeling of complete graphs K_n , where n is odd, is almost similar to construction of a magic square of order n . The construction of a magic square of order n is described in [2]. We however have to proceed differently as we do not use all the n^2 numbers as is the case with a magic square of order n . The details of our scheme are in the following theorem.

Theorem 2.1. *There is a vertex-magic labeling for K_n , for all n odd.*

Proof. It is trivial for $n = 1$. For $n > 1$, we construct the labeling as follows.

Consider an $n \times n$ matrix M . Let i, j are the indices for columns and rows respectively. The indices follow a zero based index. The process of filling each entries of the matrix is similar to constructing a magic square of order n . The only difference is that we stop filling the matrix once we have reached the number $n + \frac{n(n-1)}{2}$. The process is described below.

- **Step 1:** Let $x = 1$. Start populating the matrix from the position $j = \frac{(n-1)}{2}$, $i = \frac{(n+1)}{2}$.
- **Step 2:** Fill the current (i, j) position with the value x , and increment x by 1.
- **Step 3:** The subsequent entries are filled by moving south-east by one position (i.e. $i = (i+1) \bmod n, j = (j+1) \bmod n$), till a vacant entry is found. If the cell is already filled, then fill from the south-west direction ($i = i + 1 \bmod n, j = j - 1 \bmod n$). The value of x has to be incremented after filling an entry. If x reaches $n + \frac{n(n-1)}{2}$, then proceed to Step 4.
- **Step 4:** The remaining $n(n-1)/2$ entries are filled just by copying the non-diagonal entries by appealing to symmetry (i.e., just copying the numbers in the

lower triangular matrix to the upper triangular matrix and vice-versa from filled cells to unfilled cells).

At this stage the numbers entered along the principle diagonal are used to label the vertices of the graph from v_1 to v_n . To order the entries as given in Figure 1 we simply rearrange the entries, such that all the vertex labels come in first row i.e., move the entries along principle diagonal entries to the first row and each entry in the upper triangular matrix one position down from their previous position. \square

Lemma 2.2. *The above labeling is a valid vertex total magic labeling of K_n , n is odd. Moreover, the magic constant obtained is $S(n) = \frac{n^4+3n^2}{4n}$.*

Proof. Notice that the only difference from a magic square to our labeling matrix is that while in a magic square the entries range from 1 to n^2 , here we have used only labels from 1 to $n + n(n - 1)/2$. The other entries can be seen to be reduced in magnitude by an amount of $n + n(n - 1)/2$. Since this applies to all the entries, it can be seen that the properties of the magic square that all column sums are equal still applies in our case. Hence, the labeling obtained is a vertex magic total labeling of K_n , where n is odd.

It can be seen that the magic constant using the above construction is $K(n) = \frac{n^3+3n}{4}$. \square

Example 2.3. *We provide an example below with $n = 5$.*

The matrices in Figure 2 show the result obtained after completion of Step 3 and Step 4.

| | | | | |
|----|----|----|----|----|
| 11 | | 07 | | 03 |
| 04 | 12 | | 08 | |
| | 05 | 13 | | 09 |
| 10 | | 01 | 14 | |
| | 06 | | 02 | 15 |

| | | | | |
|----|----|----|----|----|
| 11 | 12 | 13 | 14 | 15 |
| 04 | 04 | 07 | 10 | 03 |
| 07 | 05 | 05 | 08 | 06 |
| 10 | 08 | 01 | 01 | 09 |
| 03 | 06 | 09 | 02 | 02 |

Figure 2: The resulting matrices after Step 3, on the left, and Step 4, on the right.

| | | | | | |
|----------------|----|----|----|----|----|
| Vertex label | 11 | 12 | 13 | 14 | 15 |
| Incident edges | 04 | 04 | 07 | 10 | 03 |
| | 07 | 05 | 05 | 08 | 06 |
| | 10 | 08 | 01 | 01 | 09 |
| | 03 | 06 | 09 | 02 | 02 |

Figure 3: The vertex magic total labeling for K_5 . The columns sum to 35.

We then perform rearrangement of entries to arrive at the form given in Figure 1. The resulting matrix is shown in Figure 3. \square

3 Vertex Magic Total Labeling for K_N , N is even.

In this section, we address the construction of vertex magic total labeling for K_n , when n is even. This is done in three steps. We treat the case where $n \equiv 2 \pmod{4}$ (cf. Theorem 3.1 first as in that case $n/2$ is odd. This lets us use the result of Theorem 2.1 and compose labelings of smaller graphs to arrive at the labeling of K_n . This is then extended to the case where $n \equiv 4 \pmod{8}$ and then finally where $n \equiv 0 \pmod{8}$.

In the remainder of this section, the case of $n \equiv 2 \pmod{4}$ is treated in Section 3.1, the case of $n \equiv 4 \pmod{8}$ in Section 3.2, and the case of $n \equiv 0 \pmod{8}$ is presented in Section 3.3.

3.1 Vertex Magic Total Labeling for K_n , $n \equiv 2 \pmod{4}$.

Theorem 3.1. *There is a vertex magic labeling for K_N , for all $N \equiv 2 \pmod{4}$, for $N > 2$.*

Proof. For convenience, let $n = N/2$. Our construction of vertex magic labeling for K_N makes use of a magic square of order n and a vertex magic total labeling of K_n . The construction of a magic square of order n and vertex magic total labelling for K_n is described in Section 2.

The basic idea behind constructing a vertex magic total labeling of these complete graphs is to view the graph K_N as a union of two complete graphs each of n vertices and a complete bipartite graph with n vertices on each side, i.e., $K_N = K_n \cup K_n \cup K_{n,n}$. We use the numbers 1 through $N + N(N - 1)/2$ for the labels of the vertices and edges of K_N . To arrive at the labeling, we partition this set of numbers into five disjoint sets as follows.

$$\begin{aligned} S_1 &= \bigcup_{i=1}^{n-1/2} \{(2i+1)n+1, (2i+1)n+2, \dots, (2i+2)n\} \cup \{1, 2, \dots, n\} \\ S_2 &= \bigcup_{i=2}^{n-1/2} \{2i \cdot n+1, 2i \cdot n+2, \dots, (2i+1)n\} \cup \{n+1, n+2, \dots, 2n\} \\ S_3 &= \{2n+1, 2n+2, \dots, 3n\}, \\ S_4 &= \{n^2+n+1, n^2+n+2, \dots, n^2+2n\}, \\ S_5 &= \{n^2+1, n^2+2, \dots, n^2+n\} \cup \{n^2+2n+1, n^2+2n+2, \dots, 2n^2+n\}. \end{aligned}$$

In our construction, we use sets S_1 and S_4 for constructing intermediate labeling L_1 for $K_{n/2}$. The elements of S_1 are used to label the edges and elements of S_4 are used to label the vertices. Similarly, we use sets S_2 and S_3 for constructing intermediate labelling L_2 for another $K_{n/2}$. The elements of S_2 are used to label the edges and elements of S_3 are used to label the vertices. The elements of S_5 are used to label the edges in the bipartite graph $K_{n/2, n/2}$. More detail of the construction is shown in Figure 4.

For constructing L_1 and L_2 , first consider a labelling L for n -odd. The matrices L_1 and L_2 are obtained from L as follows. Replace the vertex labels of L by the elements

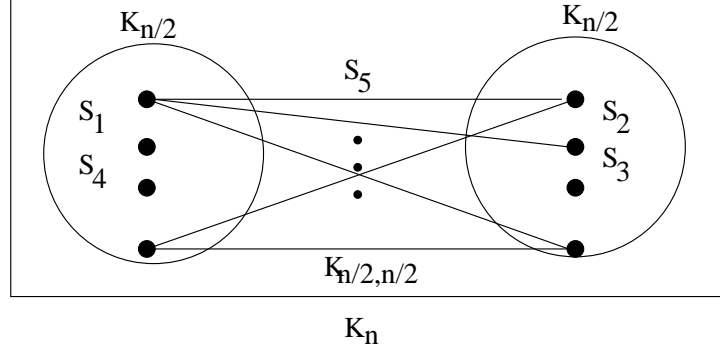


Figure 4: The construction of the labeling for K_N , with $N \equiv 2 \pmod{4}$. The two $K_{n/2}$ s and $K_{n/2, n/2}$ are shown along with the sets that are used to label the various components of K_n are also shown in the picture.

in S_4 and replace the edge labels by the elements of S_1 , to get L_1 . Similarly, if the edge labels are replaced by elements of S_2 and the vertex labels by S_3 , we get L_2 .

For constructing the labels of the edges in the bipartite graph $K_{n/2, n/2}$, consider a magic square of order $n/2$ and replace the elements of the magic square by the elements of S_5 . Let us name this modified magic square as N_1 . Transpose the magic square N_1 and name this as N_2 . Finally, arrange all these intermediate labellings as shown bellow.

$$\begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix}$$

It can be deduced that the magic constant in this case is $\frac{8n^3+6n^2-2n}{4}$. \square

Example 3.2. Let us construct vertex magic total labelling for K_6 using a vertex magic total labelling of K_3 . Let L below be a vertex magic total labeling of a K_3 , obtained by using the procedure of Section 2.

| | | | |
|-----------------------|----|----|----|
| <i>vertex label</i> | 04 | 05 | 06 |
| <i>Incident edges</i> | 03 | 03 | 02 |
| | 02 | 01 | 01 |

From this L , L_1 is obtained by using elements of S_1 and S_4 , with $S_1 = \{1, 2, 3\}$, and $S_4 = \{13, 14, 15\}$, we get the following matrix as L_1 .

| | | | |
|-----------------------|----|----|----|
| <i>Vertex label</i> | 13 | 14 | 15 |
| <i>Incident edges</i> | 03 | 03 | 02 |
| | 02 | 01 | 01 |

Similarly, L_2 is obtained by using elements of S_2 and S_4 with $S_2 = \{4, 5, 6\}$ and $S_3 = \{7, 8, 9\}$.

| | | | |
|----------------|----|----|----|
| Vertex label | 07 | 08 | 09 |
| Incident edges | 06 | 06 | 05 |
| | 05 | 04 | 04 |

To construct intermediate labelling N_1 and N_2 , consider a magic square of order 3 as follows. In Figure 5, the matrix on the left shows a magic square of order 3, the matrix on the right shows the magic square with elements replaced from elements in S_5 .

| | | |
|----|----|----|
| 04 | 09 | 02 |
| 03 | 05 | 07 |
| 08 | 01 | 06 |

| | | |
|----|----|----|
| 16 | 21 | 11 |
| 12 | 17 | 19 |
| 20 | 10 | 18 |

Figure 5: The construction of N_1 from a magic square of order 3

The matrix N_2 , for reasons of symmetry, is constructed as the transpose of N_1 . The final matrix that shows a vertex magic total labeling of K_6 , obtained by putting together L_1, L_2, N_1 , and N_2 is shown in Figure 6.

| | | | | | | |
|----------------|----|----|----|----|----|----|
| Vertex label | 13 | 14 | 15 | 7 | 8 | 9 |
| Incident Edges | 03 | 03 | 02 | 06 | 06 | 05 |
| | 02 | 02 | 01 | 05 | 04 | 04 |
| | 16 | 21 | 11 | 16 | 12 | 20 |
| | 12 | 17 | 19 | 21 | 17 | 10 |
| | 20 | 10 | 18 | 11 | 19 | 18 |

Figure 6: A vertex magic total labeling of K_6 . The first row elements are the vertex labels. The column entries of the i th column give the edge labels of edges adjacent to vertex v_i . See also the representation in Figure 1.

□

3.2 Labeling for K_N where $N \equiv 4 \pmod{8}$

We now move to the case where $N \equiv 4 \pmod{8}$. We show the following theorem.

Theorem 3.3. *There is a vertex magic labeling for K_N , for all $N \equiv 4 \pmod{8}$, for $N > 8$.*

Proof. Our construction of vertex magic labeling for K_N makes use of a magic squares of order $N/4$. Let $n = N/4$. The construction of a magic square of order n and a vertex magic total labelling for K_n is known to exist from Section 2. The basic idea behind

constructing a vertex magic total labeling of K_N , $N \equiv 4 \pmod{8}$, is to represent K_N as the union of four K_n 's, two $K_{n,n}$ s, and a $K_{2n,2n}$.

It is required to map the vertices and edges in these smaller graphs to numbers from 1 through $N + N(N - 1)/2$. To this end, we partition the integers from 1 through $N + N(N - 1)/2$ into 11 disjoint sets S_1 through S_{11} as follows.

$$\begin{aligned} S_1 &= \cup_{i=1}^{n-1/2} \{(2i+1)n+1, (2i+1)n+2, \dots, (2i+2)n\} \cup \{1, 2, \dots, n\} \\ S_2 &= \cup_{i=2}^{n-1/2} \{2i \cdot n+1, 2i \cdot n+2, \dots, (2i+1)n\} \cup \{n+1, n+2, \dots, 2n\} \\ S_3 &= \{2n+1, 2n+2, \dots, 3n\} \\ S_4 &= \{n^2+n+1, n^2+n+2, \dots, n^2+2n\} \\ S_5 &= \{n^2+1, n^2+2, \dots, n^2+n\} \cup \{n^2+2n+1, n^2+2n+2, \dots, n^2+2n\}. \end{aligned}$$

The sets S_6 , S_7 , S_8 , S_9 , and S_{10} are constructed by adding $2n^2+n$ to each element of S_1 through S_5 respectively. Finally, $S_{11} = \{4n^2+2n+1, 4n^2+2n+2, \dots, 8n^2+2n\}$ will be a set of $4n^2$ elements.

The way we use these sets is as follows. We use sets S_1 and S_4 for constructing intermediate labeling L_1 for a K_n with the elements of S_1 used to label the edges and elements of S_4 for labeling the vertices. Similarly, for labeling another K_n , we use the elements of S_2 are used to label the edges and the elements of S_3 to label the vertices. For the third K_n , we use the elements of S_6 to label the edges and the elements of S_8 to label the vertices. Similarly, for the fourth K_n , we use the elements of S_7 to label the edges and the elements of S_9 to label the vertices. In the above, the terms first K_n etc. are pictured in Figure 7.

Notice that we have to label the edges between the vertices in the first and the second K_n . For this we use the elements of S_5 . Similarly, the elements of S_{10} are used to label the edges between the third and the fourth K_n . We are now left with the edges between the first and second K_n to the third and fourth K_n . For this we use the elements of S_{11} .

For constructing L_1 through L_4 , we use a similar approach used in the proof of Theorem 3.1. We first construct a magic square of order n and replace the edge and vertex labels with the elements of the appropriate set.

We now focus on the construction of N_1 through N_4 , which serve as edge labels of the edges between the first and second K_n and edges between the third and the fourth K_n . Consider a magic square of order n and replace the entries by the elements of S_5 to get N_1 . We define $N_2 = N_1^T$. The entries in N_1 and N_2 thus serve as the labels of the edges between the first and second K_n . Similarly, we obtain N_3 by replacing the entries of a magic square of order n by the entries of S_{10} . We then define $N_4 = N_3^T$. The entries in N_3 and N_4 thus serve as the labels of the edges between the third and fourth K_n .

Finally, we show the construction of labels for the edges from the first and second K_n to the third and the fourth K_n . These labels are from the set S_{11} which has $4n^2$ numbers. We form these labels as 4 matrices M_1 through M_4 and their transposes. The matrix

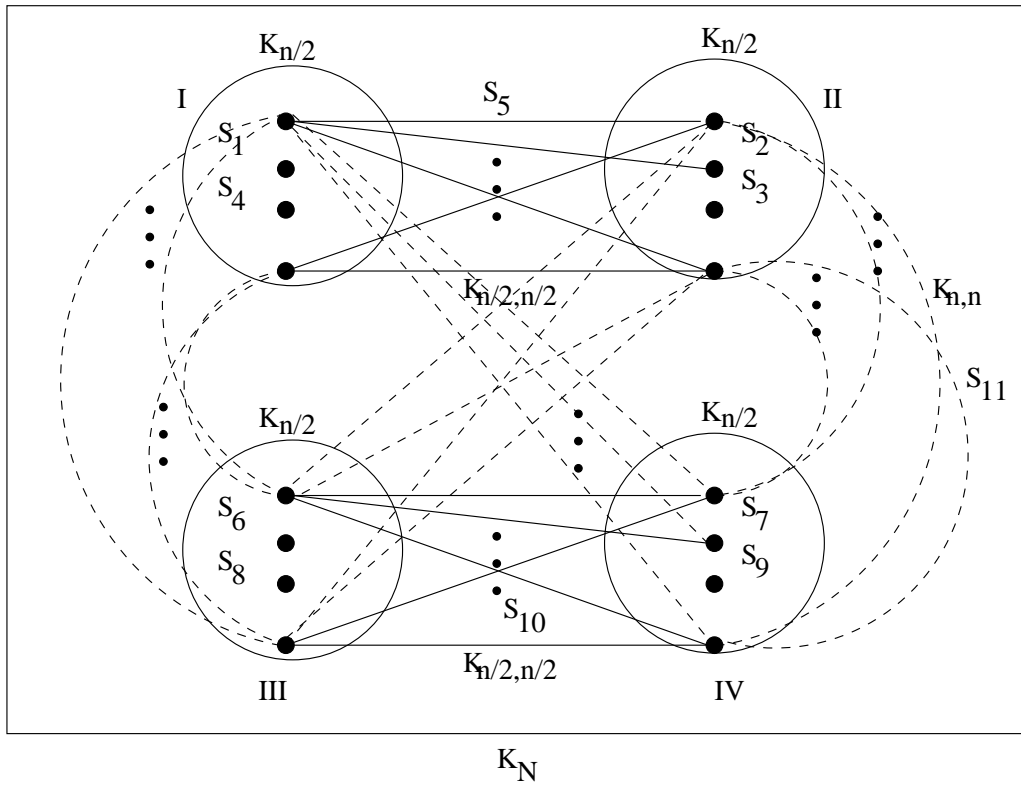


Figure 7: The construction of the labeling for K_N , with $N \equiv 4 \pmod 8$. The four smaller K_n s are labeled I, II, III, and IV. The sets that are used to label the various components of K_N are also shown in the picture. The dashed edges correspond to $K_{2n, 2n}$.

M_i , for $1 \leq i \leq 4$ is obtained by replacing the entries of a magic square of order n with the entries of S_{11} ranging from $(i-1)n^2 + 1$ to $i \cdot n^2$.

Finally, arrange all these intermediate labellings as shown below.

$$\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ N_3 & N_4 & N_1 & N_2 \\ M_1 & M_2 & M_3^T & M_2^T \\ M_4 & M_3 & M_4^T & M_3^T \end{bmatrix}$$

It can be deduced that the magic constant in this case is $\frac{32n^3+13n^2+n}{2}$. \square

Example 3.4. Let us construct vertex magic total labelling for K_{12} using a vertex magic total labeling of K_3 .

Let L , shown below, be a vertex magic total labeling of K_3 .

| | | | |
|-----------------------|----|----|----|
| <i>vertex label</i> | 04 | 05 | 06 |
| <i>Incident edges</i> | 03 | 03 | 02 |
| | 02 | 01 | 01 |

From this, the matrix L_1 is obtained by using $S_1 = \{1, 2, 3\}$ for the edge labels and $S_4 = \{13, 14, 15\}$ for the vertex labels. The resulting L_1 is shown below.

| | | | |
|-----------------------|----|----|----|
| <i>Vertex label</i> | 13 | 14 | 15 |
| <i>Incident edges</i> | 03 | 03 | 02 |
| | 02 | 01 | 01 |

Using $S_2 = \{4, 5, 6\}$, $S_3 = \{7, 8, 9\}$, $S_6 = \{22, 23, 24\}$, and $S_9 = \{34, 25, 36\}$, we obtain the matrices L_2, L_3 , and L_4 respectively as described in the proof of Theorem 3.3. The resulting matrices are shown below.

$$L_2 = \begin{bmatrix} 7 & 8 & 9 \\ 6 & 6 & 5 \\ 5 & 4 & 4 \end{bmatrix} \quad L_3 = \begin{bmatrix} 34 & 35 & 36 \\ 24 & 24 & 23 \\ 23 & 22 & 22 \end{bmatrix} \quad L_4 = \begin{bmatrix} 28 & 29 & 30 \\ 27 & 27 & 26 \\ 26 & 25 & 25 \end{bmatrix}$$

To construct the matrices N_1 and N_2 , consider a magic square of order 3 and the set $S_5 = \{10, 11, 12, 16, 17, 18, 19, 20, 21\}$. Replace the elements of the magic square by elements of S_5 to get N_1 . The resulting N_1 and N_2 are shown below.

$$N_1 = \begin{bmatrix} 16 & 21 & 11 \\ 12 & 17 & 19 \\ 20 & 10 & 18 \end{bmatrix} \quad N_2 = \begin{bmatrix} 16 & 12 & 20 \\ 21 & 17 & 10 \\ 11 & 19 & 18 \end{bmatrix}$$

Similarly, using the elements of the sets $S_{10} = \{31, 32, 33, 37, 38, 39, 40, 41, 42\}$, the matrices N_3 and N_4 are obtained as below.

$$N_3 = \begin{bmatrix} 37 & 42 & 32 \\ 33 & 38 & 40 \\ 41 & 31 & 39 \end{bmatrix} \quad N_4 = \begin{bmatrix} 37 & 33 & 41 \\ 42 & 38 & 31 \\ 32 & 40 & 39 \end{bmatrix}$$

Finally, the set S_{11} has 36 consecutive elements from 43 to 78. Of these, the first 9 elements are used to construct the matrices M_1 , the second 9 elements are used to construct M_2 , the third 9 elements for M_2 and the final 9 elements for M_4 . For purposes of brevity, we show only M_1 and M_2 in the following. The matrix M_1 is obtained as a magic square of order 3 with elements from 43 to 51.

$$M_1 = \begin{bmatrix} 46 & 51 & 44 \\ 45 & 47 & 49 \\ 50 & 43 & 48 \end{bmatrix} \quad M_2 = \begin{bmatrix} 55 & 60 & 63 \\ 54 & 56 & 58 \\ 59 & 52 & 57 \end{bmatrix}$$

The final matrix after combining all of these matrices is shown in Figure 8.

| Vertex label | 13 | 14 | 15 | 7 | 8 | 9 | 34 | 35 | 36 | 28 | 29 | 30 |
|----------------|----|----|----|----|----|----|----|----|----|----|----|----|
| Incident Edges | 3 | 3 | 2 | 6 | 6 | 5 | 24 | 24 | 23 | 27 | 27 | 26 |
| | 2 | 1 | 1 | 5 | 4 | 4 | 23 | 22 | 22 | 26 | 25 | 25 |
| | 37 | 42 | 32 | 37 | 33 | 41 | 16 | 21 | 11 | 16 | 12 | 20 |
| | 33 | 38 | 40 | 42 | 38 | 31 | 12 | 17 | 19 | 21 | 17 | 10 |
| | 41 | 31 | 39 | 32 | 40 | 39 | 20 | 10 | 18 | 11 | 19 | 18 |
| | 46 | 51 | 44 | 55 | 60 | 53 | 46 | 45 | 50 | 55 | 54 | 59 |
| | 45 | 47 | 49 | 54 | 56 | 58 | 51 | 47 | 43 | 60 | 56 | 52 |
| | 50 | 43 | 48 | 59 | 42 | 57 | 44 | 49 | 48 | 53 | 58 | 57 |
| | 73 | 78 | 71 | 64 | 69 | 62 | 73 | 72 | 77 | 64 | 63 | 68 |
| | 72 | 74 | 76 | 63 | 65 | 67 | 78 | 74 | 70 | 69 | 65 | 61 |
| | 77 | 70 | 75 | 68 | 61 | 66 | 71 | 76 | 75 | 62 | 76 | 66 |

Figure 8: A vertex magic total labeling of K_{12} . As earlier, the elements of the first row are the vertex labels and the remaining column entries are the edge labels.

□

3.3 Labeling for $N \equiv 0 \pmod{8}$

Finally, we focus on the case where $n \equiv 0 \pmod{8}$. In this case, we use mostly similar techniques as that of Theorem 3.1. Hence, we provide an outline of the proof for the following theorem.

Theorem 3.5. *There is a vertex magic labeling for K_n , for all $N \equiv 0 \pmod{8}$, for $N > 8$.*

Proof. (Outline). Notice that in this case, we can represent K_N as two $K_{N/2}$ and a $K_{N/2, N/2}$ with $N/2 \equiv 0 \pmod{4}$. We then use the constructions of Theorem 3.3 and Theorem 3.1 to combine the labels of the $K_{N/2}$ with the edge labels of $K_{N/2, N/2}$ and arrive at a vertex magic total labeling of K_N with $N \equiv 0 \pmod{8}$. \square

4 Conclusions

In this paper, we used cohesive techniques to show that every complete graph has a vertex magic total labeling. Our techniques attempt to draw connections between graph factorization and graph labeling. It is worth to explore the stronger connections between the two.

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