On Acyclic Vertex Coloring of Grid like graphs

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Abstract

d-dimensional partial tori are graphs that can be expressed as cartesian product of d graphs each of which is an induced path or cycle. Some well known graphs like d-dimensional hypercubes, meshes and tori are examples belong to this class. Muthu et al.[MNS06] have studied the problem of acyclic edge coloring for such graphs. We try to explore the acyclic vertex coloring problem for these graphs. In this respect, we provide coloring schemes to acyclically color a few basic graphs of this class such that at least one of the factor graphs is an induced cycle. Some of our schemes are optimal while others are close to optimal.

Key words: Acyclic Coloring, Graph, Acyclic Chromatic Number, Graph Product.

1 Introduction

A graph coloring is said to be *proper* if no two adjacent vertices have been assigned the same color. A coloring of a graph is said to register a *bichromatic cycle* if there exists a cycle in the graph such that all the vertices in that cycle are colored with only two colors. A vertex coloring of a graph is said to be *acyclic* if there does not exist any bichromatic cycle. In other words, the subgraph induced by any two color classes is acyclic. That is, it is a disjoint collection of trees, or a forest. The smallest number of colors needed to acyclically color the vertices of a graph is called its *acyclic chromatic number*, and is denoted by $a(G)$.

Acyclic colorings were introduced by Grünbaum[Grü73] in 1973 with a special emphasis on planar graphs. He conjectured that any planar graph can be acyclically vertex colored with 5 colors. This conjecture was later proved by Borodin[Bor06]. Determing the acyclic chromatic number is a hard problem from both a theoretical and an algorithmic point of view. More specifically, A.V. Kostochka proved in 1978 in his thesis that it is an NP-complete problem to decide for a given G and k if the acyclic chromatic number of G is at most k, even for $k = 3$ [Kos78]. It is NP-complete even when restricted to the class of bipartite graphs [CC86]. Even for highly structured class of complete graphs, the value of $a'(G)$ is still not determined. Alon and Zaks proved that it is NP-complete to determine if $a'(G) \leq 3$ for an arbitrary graph G[AZ02].

1.1 Notations

We use several notations as used in [MNS06]. We recap them here briefly. P_k is used to denote a simple path on k vertices such that $V(P_k) = \{0, \ldots, k-1\}$ and $E(P_k) = \{(i, j) : |i - j| = 1\}$.

Similarly, C_k is used to denote a cycle on k vertices such that $V(C_k) = \{0, \ldots, k-1\}$ and $E(C_k) = E(P_k) \cup (k-1, 0)$. PATHS denotes the set $\{P_2, P_3, \ldots\}$ of all paths on 2 or more vertices. Similarly, CYCLES denotes the set $\{C_3, C_4, \ldots\}$ of all cycles. Rest of the notations are standard graph theory notations [Wes01], such as a K_n represents a complete graph on n vertices.

1.2 Graph Factorization

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the cartesian product of G_1 and G_2, denoted by $G_1 \Box G_2$, is defined to be the graph $G = (V, E)$ where $V = V_1 \times V_2$ and E contains the edge joining (u_1, u_2) and (v_1, v_2) if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E_1$.

Note that the graph product operation is commutative i.e. $G_1 \Box G_2$ and $G_2 \Box G_1$ are isomorphic. They are similarly associative. Hence, the graph $G_1 \Box \cdots \Box G_d$ is unambiguously defined for any d. G^d denotes the d-fold cartesian product of a graph with itself. Sabisussi[Sab59] and Vizing[Viz63] have shown that any connected graph G can be expressed as a product $G_1 \Box \cdots \Box G_k$ of primes factors $G_i (1 \leq i \leq k)$. A graph is said to be *prime* with respect to the $\Box$ operation if it has at least two vertices and it is not isomorphic to the product of two non-trivial graphs (having at least two vertices). Also, this factorization is unique except for a re-ordering of the factors and is referred to as the *Unique Prime Factorization* of the graph. Since $a(G)$ is a graph invariant, we assume without loss of generality that any graph that is either an induced path or an induced cycle has all the factors from PATHS $\cup$ CYCLES.

Notice that when for a graph G, each of the factors $G_i = P_2$, then G is a d-dimensional hypercube. Similarly, when each of the $G_i \in$PATHS, then G is a d-dimensional mesh, and, when every $G_i \in$CYCLES, then G is a d-dimensional torus.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two families of graphs. Then, we define an operation PROD between $\mathcal{G}_1$ and $\mathcal{G}_2$ such that

$$\mathcal{G}_1 \text{ PROD } \mathcal{G}_2 = \{G_1 \Box G_2 | G_1 \in \mathcal{G}_1 \text{ and } G_2 \in \mathcal{G}_2\}$$

1.3 Our Results

We give coloring schemes to acyclically vertex color certain basic graphs of the d-dimensional partial tori family. We have considered graphs that have at least one factor as a member of CYCLES. We first provide coloring scheme for a special case $C_m \Box P_2$ and then extend it to the general $C_m \Box P_k$ (k>2). All these schemes are optimal i.e. we prove the acyclic chromatic number for these graphs to be 3 and 4 respectively. We then provide a coloring scheme for a graph that has all members from CYCLES. In this respect, we give a scheme to acyclically vertex color $C_m \Box C_k$ that uses 7 colors. Since such a graph will use at least as many colors as a $C_m \Box P_k$, our results can differ from the optimal value by at most 3.

1.4 Related Work

Fertin et al.[FGR03] have explored the acyclic vertex coloring of grids which are a subclass of the more general d-dimensional partial tori graphs. In other words, none of the prime factors of the graphs that they consider belongs to CYCLES. They describe lower and upper bounds for acyclic chromatic number of d-dimensional grid $G(n_1, \ldots, n_d)$. $G(n_1, \ldots, n_d)$ denotes that G has n_i vertices in the ith dimension ($1 \leq i \leq d$). They also show that these bounds match, and thus give an optimal result, when the length in each dimension is *sufficiently large*, or more precisely, if $\sum_{i=1}^{d} \frac{1}{n_i} \leq 1$. If it is not the case, then these bounds differ by an additive constant of at most $|1 - \lfloor \sum_{i=1}^{d} \frac{1}{n_i} \rfloor|$. Also, they apply those results to a hypercube of d dimensions, which is a special case of $G(n_1, \ldots, n_d)$ in which there are only 2 vertices in each dimension. In this case, their bounds differ by a multiplicative constant of 2.

Muthu et al. [MNS06] have explored the acyclic edge coloring for d-dimensional partial tori graphs. They provide several results on the change in the acyclic edge chromatic number when a graph is taken product with a member of either PATHS or CYCLES.

2 Coloring Schemes - PATHS PROD CYCLES

We consider the graphs where a member of CYCLES is taken product with a member of PATHS. Notice that this leads to forming multiple copies of the operand cycle that are connected with corresponding vertices in the copies. The number of such copies being equal to the length of the operand path; when taken product with another member of CYCLES, this is equal to the length of the second operand. This gives us the advantage of symmetry in the manner that any scheme that works within a cycle works for all the copies. So, we focus only on the edges connecting them. Each of the following cases describes this further.

2.1 $C_m \Box P_2, m$ is odd

We consider the coloring of the graphs produced when a cycle of odd length C_m (m is odd), is taken product with a P_2. Figure 1 shows this resulting graph. As observed before, we can hope to modify the acyclic coloring of the cycle and apply it to the copy resulting in the acyclic vertex coloring of the whole graph.

Lemma 1. *$C_m \Box P_2$ can be acyclically vertex colored with 3 colors for all odd m.*

Proof. First of all, we note that $a(C_m) = 3$ for all $m > 2$. So, let us assume without loss of generality, that the operand C_m was colored with 3 colors as shown in figure 4.1. Clearly this given scheme acyclically vertex colors C_m. Now, let us denote the copy as C'_m. We propose to modify this scheme to suitably color C'_m as follows.

If v is a vertex in the C_m, then let $\sigma(v)$ denote the corresponding vertex in C'_m. Then, if $c(v)$ denotes the color of v, then

$$c(\sigma(v)) = \{c(v) \bmod 3\} + 1 \tag{1}$$

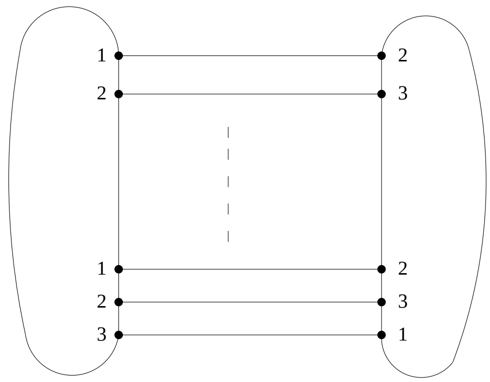

Figure 1: A $C_m \square P_2$, where m is odd

Now, we see that such a vertex coloring is proper as well as acyclic for the product graph. Clearly, this coloring is proper since there are no adjacent vertices with the same color. This is because

1. C_m is given to be properly colored. So, C'_m is properly colored as well.
2. $c(\sigma(v)) \neq c(v)$ for any v.

To show that this coloring is acyclic we consider all the possible bichromatic cycles that can occur. Also note that since both C_m and C'_m are acyclically colored, any such cycle should involve vertices from both of them.

1. $1-2\cdots-1$ *cycle* This cannot happen since the only 1-colored vertex in C'_m does not have two neighbors with color 2.
2. $2-3\cdots-2$ *cycle* The only 3-colored vertex in C_m does not have two neighbors with color 2. So, this cannot occur as well.
3. $3-1\cdots-3$ *cycle* None of the 1-colored vertices in C_m have two 3-colored neighbors.

Hence, this graph can be colored with 3 colors which is also optimal. Thus, the acyclic chromatic number of the graph, $C_m \square P_2$ is 3 when m is odd. □

2.2 $C_m \square P_2, m$ is even

Lemma 2. *The acyclic chromatic number of $C_m \square P_2$ is 3 i.e. $a(C_m \square P_2) = 3$ for all $m \neq 4$.*

Proof. In this case we assume that the given C_m is colored with 3 colors as shown in figure 2. Such a coloring is clearly acyclic. Notice how the last four vertices of the cycles have been colored to make it look similar to the last case to reuse the conditions that prevent a bichromatic cycle. We also note that this requires m to be at least 6. Now we try to color the product graph based on this.

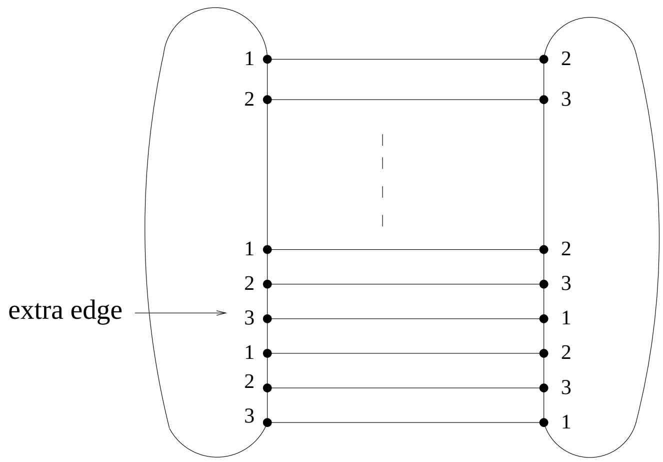

Figure 2: A $C_m \Box P_2$, where m is even

As before, we denote the copy of the operand C_m as C'_m and we define a function σ that maps the colors from the C_m to C'_m. In fact, we use the same σ as before i.e.

$$c(\sigma(v)) = \{c(v) \bmod 3\} + 1 \tag{2}$$

Clearly, this coloring is proper by the same argument as before. To show that it is also acyclic, we again go through all possible bichromatic cycles. We also mention here that since both C_m as well as C'_m are acyclically colored, any such cycle must involve both of them.

1. $1 - 2 \cdots - 1$ *cycle* This cannot happen since the only 1-colored vertex in C'_m does not have two neighbors with color 2.

2. $2 - 3 \cdots - 2$ *cycle* The only 3-colored vertices in C_m do not have two neighbors with color 2. So, this cannot occur as well.

3. $3 - 1 \cdots - 3$ *cycle* None of the 3-colored vertices in C'_m have two neighbors that are 1-colored.

Hence, this coloring scheme optimally colors the given graph with 3 colors. By clubbing this result with the previous one, we can state that the acyclic chromatic number of $C_m \Box P_2$ is 3 for all $m \neq 4$. $\square$

2.3 $C_m \Box P_k$

Now we extend our previous results to the case of $C_m \Box P_k$ (k>2), i.e. we now have more than just two copies of the operand C_m. We color alternate copies of C_m with same color i.e. all odd copies with one color scheme and all even copies with another such that the whole graph gets colored acyclically. Figure 3 shows such a coloring.

Lemma 3. *$C_m \Box P_k$ can be acyclically vertex colored with 4 colors for all m and $k > 2$.*

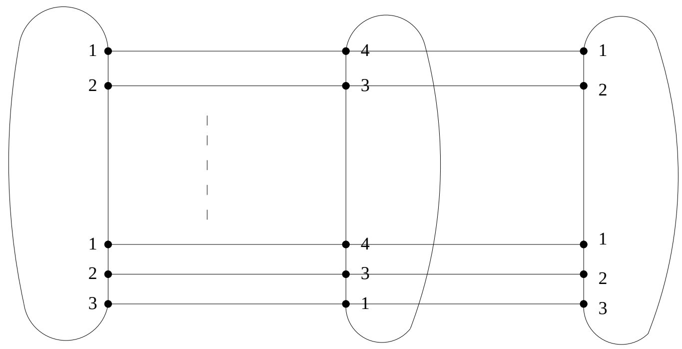

Figure 3: A $C_m \Box P_3$ colored with 4 colors, where m is odd

Proof. We formally define this coloring scheme by providing a mapping function from the colors in the operand C_m to corresponding vertices in the even copies. As before, we define such a function that maps the colors of operand C_m to the even copies as following.

$$
\begin{aligned}
c(\sigma(v)) &= 4 \text{ if } c(v) = 1 && (3)\\
&= 3 \text{ if } c(v) = 2 && (4)\\
&= 1 \text{ if } c(v) = 3 && (5)
\end{aligned}
$$

Clearly, this coloring is proper. This is also acyclic because the only possible bichromatic cycles should involve at least two copies of the operand C_m and none of the color pairs can form a cycle that can *come back* to the copy where is emanates after leaving it once. $\square$

2.4 Optimal Coloring of $C_m \Box P_k$

Lemma 4. *Acyclic chromatic number of $C_m \Box P_k$ is 4 i.e. $a(C_m \Box P_k) = 4$.*

Proof. Lemma 3 says that a $C_m \Box P_k$ can be colored with 4 colors. Now, we prove that this is also the optimal number of colors required to acyclically color it.

Claim 1. *$C_4 \Box P_k$ cannot be acyclically vertex colored with 3 colors for $k \geq 2$.*

To prove *Claim 1*, we use the simplest member of this family, $C_4 \Box P_2$ as shown in figure 4.

Here note that, $\{1, 2, 3, 2\}$ and $\{1, 2, 1, 3\}$ are the only possible color patterns of 3 colors that can be used to color a C_4 such that it is properly and acyclically colored. Any other color combination would just be a renaming of the colors. Also, the possible manners in which these colors can be validly mapped to the second cycle such that they are also properly and acyclically colored are shown in figure 4.

By looking at figure 4, one can see that there exists a bichromatic cycle of colors 2 and 3 in graph (i). Similarly, there is a bichromatic cycle of colors 1 and 2 in both graphs (ii) and (iii),

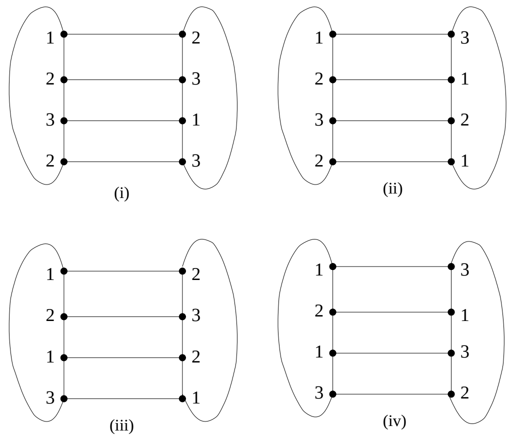

Figure 4: All possible distinct color combinations for $C_4 \square P_2$ using 3 colors

and, of colors 1 and 3 in graph (iv).

Hence, we conclude that a $C_4 \square P_2$ requires at least 4 colors to be acyclically colored. Or in other words, there exist a member of $C_m \square P_k$ requires at least 4 colors to be acyclically colored, thereby proving *Lemma 4*. $\square$

3 Coloring Scheme - CYCLES PROD CYCLES

Now we consider the graphs when a member of CYCLES is taken product with another member of CYCLES. Note that they have the same structure as $C_m \square P_k$ except that the first and last copies are connecting as well. Hence, if we color the last copy of C_m with colors that are different altogether from the ones used in other copies, then such a scheme would acyclically color $C_m \square C_k$.

Lemma 5. *All $C_m \square C_k$ can be colored with 7 colors.*

Proof. Also, we know that any C_m can be acyclically colored with 3 colors. This gives us our first result on the upper bound for number of colors to be used to color $C_m \square C_k$ acyclically.

$$a(C_m \square C_k) \leq a(C_m \square P_k) + 3 \tag{6}$$

Hence, $C_m \square C_k$ $(m, k > 2)$ can be colored with 7 colors. Figure 5 shows implementation of such a scheme. $\square$

4 Conclusion

We expect that our work for the cases of graphs of the class d-dimensional partial tori can be extended to generalised results i.e. bounds on acyclic chromatic number of type $G \square P_2$, $G \square P_k$

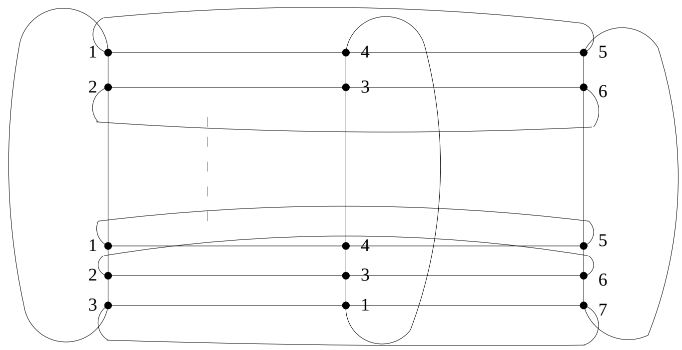

Figure 5: A $C_m \square P_3$ colored with 7 colors, where m is odd

$(k > 2)$ and $G \square C_k$ for any given graph G.

We also note that our coloring scheme for $C_m \square C_k$ seems to use the least number of colors possible. It is therefore expected that our bound might turn out to be the optimal bound for these graphs.

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